

PROBLEM 3.6.6 FROM THE TEXT

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

Question 1. Solve $y'' + 9y = 9 \sec^2 3t$ on the interval $0 < t < \frac{\pi}{6}$.

Strategy. We solve the corresponding homogeneous ODE $y'' + 9y = 0$ for the fundamental set of solutions involving $y_1(t)$ and $y_2(t)$. Then we use the method of Variation of Parameters with $Y(x) = u_1(t)y_1(t) + u_2(t)y_2(t)$ to find a particular solution to the original, nonhomogeneous ODE.

Solution. Here, it is readily apparent that a fundamental set of solutions to $y'' + 9y = 0$ is

$$c_1 \cos 3t + c_2 \sin 3t.$$

Indeed, the characteristic equation of this homogeneous, second-order, linear ODE with constant coefficients is $r^2 + 9 = 0$, with solutions $r = \pm 3i$. With the real part 0 and the imaginary part 3, the result follows.

Now, with this assumed form for $Y(t)$, we get the system

$$u_1' \cos 3t + u_2' \sin 3t = 0 \quad (1)$$

$$-3u_1' \sin 3t + 3u_2' \cos 3t = 9 \sec^2 3t. \quad (2)$$

Upon multiplying the first equation by $3 \sin 3t$ and the second by $\cos 3t$ and adding, we obtain the new equation

$$3u_2' \sin^2 3t + 3u_2' \cos^2 3t = 9 \sec^2 3t \cos 3t = 9 \sec 3t.$$

Thus $u_2' = 3 \sec 3t$ so that

$$u_2(t) = \ln |\sec 3t + \tan 3t| + C,$$

although we will ignore the constant.

Substituting this back into Equation ??, we get

$$u_1' \cos 3t + u_2' \sin 3t = 0 = u_1' \cos 3t + 3 \sec 3t \sin 3t, \text{ or } u_1' = -\frac{3 \sin 3t}{\cos^2 3t} = -3 \tan 3t \sec 3t.$$

Thus, $u_1(t) = -\sec 3t$.

Thus our particular solution to the ODE is

$$\begin{aligned} Y(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= -\sec 3t \cos 3t + \ln |\sec 3t + \tan 3t| \sin 3t \\ &= -1 + \ln |\sec 3t + \tan 3t| \sin 3t. \end{aligned}$$

And our general solution to the ODE is

$$y(x) = c_1 \cos 3t + c_2 \sin 3t - 1 + \ln |\sec 3t + \tan 3t| \sin 3t.$$

Now to check whether this particular solution $Y(t)$ is correct as a solution to the nonhomogeneous ODE:

With $Y(t)$ as above, we first calculate the two derivatives:

$$Y(t) = -1 + \ln |\sec 3t + \tan 3t| \sin 3t,$$

$$Y'(t) = 3 \sec 3t \sin 3t + 3 \ln |\sec 3t + \tan 3t| \cos 3t$$

$$= 3 \tan 3t + 3 \ln |\sec 3t + \tan 3t| \cos 3t,$$

$$Y''(t) = 9 \sec^2 3t + 9 \sec 3t \cos 3t - 9 \ln |\sec 3t + \tan 3t| \sin 3t$$

$$= 9 \sec^2 3t + 9 - 9 \ln |\sec 3t + \tan 3t| \sin 3t.$$

Now, we place these derivatives back into the original ODE and see

$$Y''(t) + 9Y(t) = 9 \sec^2 3t$$

$$(9 \sec^2 3t + 9 - 9 \ln |\sec 3t + \tan 3t| \sin 3t) + 9(-1 + \ln |\sec 3t + \tan 3t| \sin 3t) = 9 \sec^2 3t$$

$$9 \sec^2 3t = 9 \sec^2 3t.$$