HOMEWORK SET 3. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
PROFESSOR RICHARD BROWN

1. General Information

The homework sets are listed here:

`http://www.mathematics.jhu.edu/brown/s10/SyllabusS10421.htm`

2. Selected Exercises

Exercise (2.3.2). Since $f : [0,1] \to [0,1]$ is non-increasing, for $x_0, y_0 \in [0,1]$, if $y_0 > x_0$, then $f(y_0) \le f(x_0)$. Rewritten, we have for $y_0 > x_0$, $f(y_0) = y_1 \le x_1 = f(x_0)$. And since $x_1 \ge y_1$, we get $f(x_1) = x_2 \le y_2 = f(y_1)$. Put these together to get

$$\text{if } y_0 > x_0, \text{ then } f\left(f(y_0)\right) \ge f\left(f(x_0)\right).$$

That is, the map $f^2 : [0,1] \to [0,1]$ is a non-decreasing map. By Proposition 2.3.5, all $x \in [0,1]$ are either fixed points or asymptotic to fixed points. This automatically rules out any higher order periodic points for f^2 (Why?). Thus, f^2 can have only fixed points, which means that f can periodic points of order at most 2.

Example. Let $f(x) = 1 - x$. Here all points are of order-2 except for $x = \frac{1}{2}$. Also for $g(x) = 1 - x^2$, the end points are of period 2 and the only fixed point is $\frac{\sqrt{5}-1}{2}$. What can you say about the dynamics of $g(x)$?

Exercise (2.4.6). This is really just basic ODE theory: If an autonomous, first order ODE $\dot{x} = f(x)$ has an isolated equilibrium solution at x_0, and $f'(x_0) < 0$, then the equilibrium solution is a sink. The reasoning goes that for any $x < x_0$ but sufficiently close, the unique solution $x(t)$ passing through x at time-0 will always have $\dot{x}(t) > 0$ for all $t > 0$, since here $f(x) > 0$ for $x < x_0$ and close by. On the other side of x_0, at least nearby, $\dot{x} = f(x) < 0$. Hence solutions that start on this side of x_0 will decay to x_0.

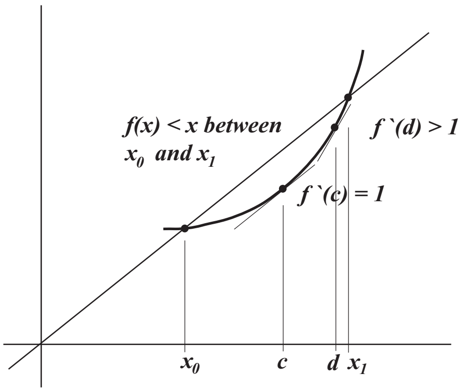
In a more general sense, a system of first order ODEs $\dot{\vec{x}} = f(\vec{x})$ which is "nice" near a fixed point $\vec{x}_0$ has a special quality near that fixed point. This special quality is that, under certain non-degeneracy conditions, the behavior of solutions on the original nonlinear system is qualitatively the same as that of the linear system formed by the first Taylor approximation to the right hand side f at the point $\vec{x}_0$ (this is the essence of the term *local linearization.* The term "nice" can be defined as "$f(x)$ is continuously differentiable in a neighborhood of x_0). For a 1×1 system, the original equation can be linearized around x_0, since $f'(x_0)$ exists, and as long as $f'(x_0) \neq 0$, the solution near x_0 of the original equation will behave much like that of the linear equation.

Given that discussion, the linearized equation is

$$\dot{x} = f'(x_0)x = kx, \quad \text{where } k < 0.$$

Solutions to this equation are $x(t) = Ce^{kt}$, and with $k < 0$, and all nearby solutions exponentially decay to 0 exponentially. Thus the original system has the same property asymptotically at x_0. Indeed, this linear system has a linear time-1 map $g : \mathbb{R} \to \mathbb{R}$, $g(x) = e^k x$. This is obviously a e^k-contraction since $k < 0$. Since the vector field $f(x)$ is continuously differentiable, the original ODE will be a e^k-contraction asymptotically at x_0. Hence, there will assuredly be a small interval around x_0 where the ODE will remain a contraction. Hence, on this interval, all solutions will converge exponentially to x_0.

Exercise (2.5.3). Suppose the fixed point set $Fix(f)$ is not connected. Then there must be a gap between two points in the fixed point set where no other fixed points reside. Call this gap the open interval (x_0, x_1) where $x_0, x_1 \in Fix(f)$ $x_1 - x_0 > 0$ (assuming $x_1 > x_0$) and $\forall x \in (x_0, x_1)$, $x \notin Fix(f)$.



Then it must be the case that for all $x \in (x_0, x_1)$, either $f(x) > x$ or $f(x) < x$ (the graph of f either lies completely above the diagonal in the unit square, or completely below it). Suppose for instance, $f(x) < x$, for all $x \in (x_0, x_1)$. By the Mean Value Theorem, there must exist a point $c \in (x_0, x_1)$ where

$$f'(c) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1 - x_0}{x_1 - x_0} = 1.$$

However, now consider the interval (c, x_1). Again by the Mean Value Theorem, there must exist a point $d \in (c, x_1)$, where

$$f'(d) = \frac{f(x_1) - f(c)}{x_1 - c} > \frac{x_1 - c}{x_1 - c} = 1$$

since here $f(c) < c$. But this is impossible, because by supposition, on all of $[0, 1]$, we have $|f'(x)| \leq 1$. Thus, by contradiction, there can be no gap between the fixed points x_0 and x_1 where the graph of f lies completely below the diagonal.

The proof in the other case, where $f(x) > x$ for all $x \in (x_0, x_1)$ is entirely symmetric and is left to you.

Exercise (2.5.4). First, some general information. Due to the previous problem, the fixed point set of any map of this kind, must be connected, so either a point or a (closed) interval. There are many reasons why it must be closed (include the endpoints), but think about why. And second, if the fixed point set is an interval, then it must be the case that the derivative along the fixed point interval is definitely 1. Draw a picture if you do not see this.

Now, to solve this problem, we will work in cases. First, let's suppose that our function $f(x)$ on $I = [0,1]$ has $|f'(x)| < 1$ and the inequality is strict. Then $Fix(f)$ consists entirely of one point, say $x_0 = Fix(f)$ by Exercise 2.5.3. In the event that $f'(x) > 0$, it should be obvious that x_0 is an attractor, at least nearby (draw a picture: with these conditions, nearby orbits that start on either side of the fixed point will stay on that side, and the orbits nearby will all be monotonic, tending toward the fixed point. As there are no other fixed points around, they must tend toward x_0. In the case that $f'(x_0) < 0$, we may have orbits that jump from one side of x_0 to the other side and then back again. However, looking at $f^2(x)$,

$$\frac{d}{dx}f^2(x) = f'\left(f(x_0)\right)f'(x_0) = [f'(x_0)] > 0.$$

Then by the same argument as above, x_0 will again be an attractor.

Do you see how important this is? Suppose there exists a periodic point $y_0 < x_0$ of of minimal period $n > 1$. Then $y_0 \in Fix(f^n)$. But also $x_0 \in Fix(f^n)$. Since by Exercise 2.5.3, fixed point sets of maps of I with derivatives bounded by 1 in absolute value must have connected fixed points, this says that the entire interval $[y_0, x_0] \subset Fix(f^n)$. This is true here because for any $n \in \mathbb{N}$, the map f^n also has the SAME bound on its derivative (can you see this?). As we have already established that x_0 is an attractor, there can be NO periodic points in a small enough neighborhood of x_0. This contradiction establishes the result in the case that the derivative of f is never equal to 1 or -1.

Now suppose that $f'(x_0) = 1$. Then either $Fix(f)$ is either just the point x_0, or includes an entire interval containing x_0. Either way, the edges of this interval will have attractors on the sides outside of the interval of fixed points. Again, draw a picture to see this. The same argument above will establish the same contradiction.

Finally, suppose $f'(x_0) = -1$. Again, then $x_0 = Fix(f)$. If you look toward f^2, $\frac{d}{dx}f^2(x_0) = 1$. It may now be the case that $Fix(f^2)$ IS an interval containing x_0 (picture the map $f(x) = 1 - x$ on $[0,1]$). In fact, the interval of fixed point of f^2 in this case is precisely the image of the interval containing x_0 where the derivative of f is precisely -1. But the above argument will again show that ther can be NO other periodic point besides these possible order-2 points. We are done.

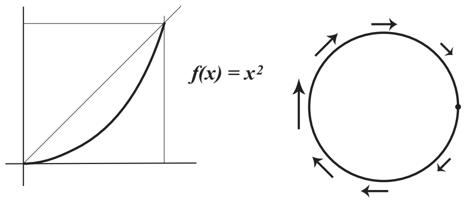
Exercise (EP9). Let $I = [a,b]$. For part a), make the assumption that the invertible map $f : I \to I$ is not one-to-one (injective). Then $\exists x, y \in I$ such that $x \neq y$ but $f(x) = f(y)$. Since $f(x)$ is invertible, f^{-1} is a function. But then

$$x = f^{-1}\left(f(x)\right) = f^{-1}\left(f(y)\right) = y$$

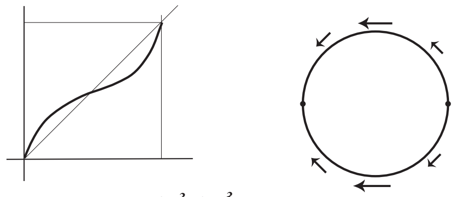
is a contradiction. Hence the assumption must be wrong. The other two parts are constructed similarly.

Exercise (EP10). To answer the first request, take any increasing function $f : I = [0,1] \to I$ that fixes both endpoints, and you are done. This will correspond to an injective continuous function on S^1. Note the fact that it is increasing allows that the inverse exists, and the fact that the endpoints are fixed makes the inverse continuous. For an example, the function $f(x) = x^2$ is a continuous invertible function on I which induces a continuous, invertible function on S^1 (What is an expression for the inverse?) See the figure:

Really, the end point need not be fixed, as long as $f(0) = f(1)$. For a differentiable circle map, the one-sided derivatives of the map f at the endpoints must

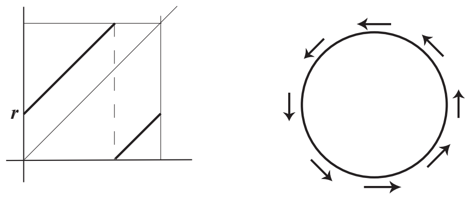


agree in addition to the map being continuous. Try $f(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}$, as in the next figure:



$f(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}x$

Here, considered as a function on $\mathbb{R}$, $f'(0) = f'(1)$. Finally, since the graph of $f(x)$ on I need not actually be in I due to the fact that we are identifying the endpoints of I, it is quite easy to create a differentiable function on the circle that comes from a function whose domain is I but whose range is shifted a bit. This leads to an easy to describe differentiable fixed point free map on the circle $f(x) = x + r$, where $r \in I$, as in the figure:



$f(x) = x + r$

Note that this only works due to the fact the via the identification of the endpoints, one can "wrap" the graph vertically. You will see these kind of graphs again in circle , cylinder and toral maps later.

Exercise (EP11). The general solution to this uncoupled system is

$$\begin{aligned} r(t) &= \frac{r_0}{r_0 + (1 - r_0)e^{-\frac{t}{2}}} \\ \theta(t) &= t + \theta_0 \\ z(t) &= z_0 e^{-t}. \end{aligned}$$

Along the circle given by the two equations $r = 1$ and $z = 0$ (remember from Calculus III that the typical 1-dimensional set in $\mathbb{R}^3$ requires two equations), the motion is constant velocity. More precisely, the solution passing through the initial point $\vec{x} = (r_0, \theta_0, z_)) = (1, 0, 0)$ is given by

$$r(t) = 1, \quad \theta(t) = t, \quad z(t) = 0.$$

Since by observation, there are no other cycles near this one (or anywhere else, for that matter), this is an isolated cycle, with period 2π.

Choose the plane passing through the point $(1, 0, 0)$, and normal to the motion along the cycle; the tangent vector to the motion passing through $(1, 0, 0)$ is the tangent vector $(0, 1, 0)$. Note that with cylindrical coordinates, we only view this plane as a half-plane, to distinguish it from the plane as that described by the equation $\theta = \pi$. Call this plane $\mathcal{P}_\theta$.

Every orbit that starts in this plane (even the origin) intersects this plane again at $t = 2\pi$. Using (r, z) as coordinates on $\mathcal{P}_\theta$, we get a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ (actually it is a function on the half-plane defined by $r \geq 0$ and z, but near the point $(r, z) = (1, 0)$, this will not matter) defined by the condition that $(r_0, z_0) = (r(0), z(0)) \mapsto (r(2\pi), z(2\pi))$. Thus

$$f(r, z) = \left(\frac{r}{r + (1 - r)e^{-\pi}},\ ze^{-2\pi} \right).$$

Notice that the point $(1, 0)$ is fixed by f and that f is nonlinear (in r only, though). However, to understand the qualitative behavior of f near the fixed point at $(1, 0)$, we calculate the derivative map there:

$$Df_{(1,0)} : \left(\begin{array}{cc} \frac{\partial f_r}{\partial r} & \frac{\partial f_r}{\partial z} \\ \frac{\partial f_z}{\partial r} & \frac{\partial f_z}{\partial z} \end{array} \right)\Bigg|_{(1,0)} = \left(\begin{array}{cc} e^{-\pi} & 0 \\ 0 & e^{-2\pi} \end{array} \right).$$

It is immediate that all of the eigenvalues of the fixed point $(1, 0)$ under the map f are positive and strictly less than 1. Hence this orbit is an attractive orbit and a limit cycle.

Exercise (EP12). Really, for λ outside of the interval $[0, 4]$, the range of f as a real-valued function includes points outside of $[0, 1]$.