

HOMework SET 2. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
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1. GENERAL INFORMATION

The homework sets are listed here:

<http://www.mathematics.jhu.edu/brown/courses/s11/Syllabus421.htm>

First, I want to establish a general claim used in the book.

Claim. For $x \in I = [1, \infty)$ and $t \geq 0$, $\sqrt{x+t} \leq \sqrt{x} + \frac{1}{2}t$.

Note. There is a typo in the book when they establish this claim (can you find it?) Here we show this with a little bit of Calculus II. Really, this only involves a statement that the tangent line to the graph of $\sqrt{x}$ lies above the graph for $x \in I$.

Proof. Let $g(t) = \sqrt{x+t}$ for some fixed $x \in I$ defined on $t \geq 0$ (really, we need $t \in (-\epsilon, \infty)$ for a small $\epsilon > 0$ since we will need the differentiability of g at $t = 0$).

Expand $g(t)$ about $t = 0$ as a Taylor Series:

$$g(t) = g(0) + g'(0)t + \frac{g''(0)}{2}t^2 + \dots$$

Here, all of the derivatives of $g(t)$ at 0 are defined and positive, so for $t \geq 0$, $g(t) \leq g(0) + g'(0)t$ (this is what I mean by the tangent line lying *over* the graph). Hence

$$\sqrt{x+t} = g(t) \leq g(0) + g'(0)t = \sqrt{x} + \frac{1}{2\sqrt{x}}t \leq \sqrt{x} + \frac{1}{2}t,$$

since $\sqrt{x} \geq 1$ on I . □

2. SELECTED EXERCISES

Exercise (2.2.5). There are a number of ways to find the second-order recursion that defines the population of lemmings. For one, start from the bottom and work your way up. Consider it this way: Take you census every spring. Then last year's summer litter is maturing and the 2 year-olds have just recently died. In year 1, we have just the one pair, and they are preparing to have young. They will have 4 young and the next winter, none will die. The next spring (year 2), there are 6 now, but the original pair is turning 2 this summer. They all produce, and the 12 new young make the fall population 18. The original 2 die in the winter, and the next spring (year 3), there are 16. More precisely, let y_n = spring population in year n , b_n the summer population 3 months later, and d_n the winter deaths 6 months after that. Then it is easy to see that

$$y_{n+1} = y_n + b_n - d_n.$$

However, there are relationships between these variables. For one, every spring pair produces four young in the summer. Thus

$$b_n = \frac{1}{2}y_n \cdot 4 = 2y_n.$$

Next, the number of deaths in any winter is precisely the number of births 2 summers before the previous spring (lemmings die in the winter of their third year). Thus

$$d_n = b_{n-2}.$$

Thus, up to now, we get

$$\begin{aligned} y_{n+1} &= y_n + 2y_n - b_{n-2} \\ &= 2y_n + (y_n - b_{n-2}). \end{aligned}$$

The last line is written this way because it is also true that the population in any spring minus the births two years before is precisely the number of births in the intermediate year, which is precisely twice the spring population of that year. Indeed,

$$y_n - b_{n-2} = b_{n-1} = 2y_{n-1}.$$

Putting this in for the term in the parentheses, we get

$$y_{n+1} = 2y_n + 2y_{n-1}$$

and we are done.

Notice that the book is incorrect in its solution to this problem. However, the rest of the problem is fine. Let $x_{n+1} = \frac{y_{n+1}}{y_n}$. Recall that the growth of a second order recursive sequence is not quite exponential, but asymptotically exponential, with a growth rate given by the limit of the sequence $\{x_n\}$. To find this, notice that

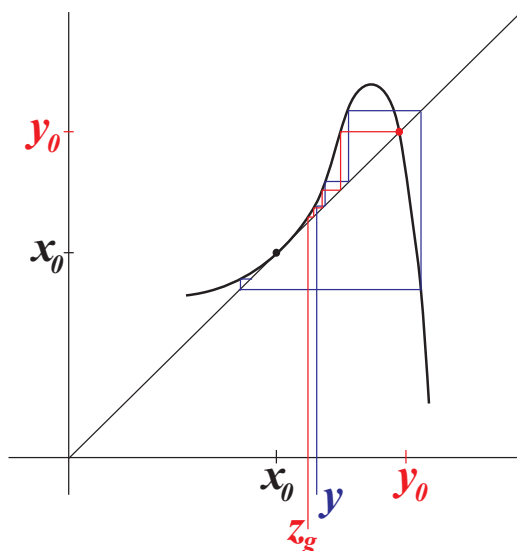
$$x_{n+1} = \frac{y_{n+1}}{y_n} = \frac{2y_n + 2y_{n-1}}{y_n} = 2 + 2\frac{y_{n-1}}{y_n} = 2 + \frac{2}{x_n}.$$

If we let $g(x) = 2 + \frac{2}{x}$, then $x_{n+1} = g(x_n)$. This is a contraction on a suitable interval like $I = [2, 4]$, and all orbits are asymptotic to the fixed point at $x = 2 + \frac{2}{x}$, which is $x^2 - 2x - 2 = 0$ (notice this similar structure of this last equation and the recursive y -sequence that defined it). The only root to this in I is $x = 1 + \sqrt{3}$.

One final note here; there are other ways to do this. One is to count lemmings each year by age. To do this every spring, simply note that every spring only the 0-year olds and the 1-year olds are to be counted, because the two year olds in that spring are actually $2\frac{3}{4}$ years of age, and have died before getting there. To get the living lemming counts, just follow the number of births each year. You still have to work out the recursion formula, but the sequence is simple to calculate.

Exercise (2.2.6). The conditions here stipulate that x_0 is a fixed point with a unit slope in absolute value and that x_0 is not an inflection point or part of an interval of fixed points, so $f''(x_0) \neq 0$. One case of this is $f'(x_0) = 1$ (the graph of $f(x)$ is tangent to the fixed point line ℓ (the line $y = x$)). One conclusion one can immediately draw is that near x_0 (except at x_0), either the graph of f is completely above ℓ or completely below it (draw pictures). Indeed, consider the former. Let $U(x_0) = B_\epsilon(x_0)$ be a small neighborhood around x_0 , where $\forall y \in U$, $y \neq x_0$, we know $f(y) > y$. But then for $y > x_0$, we have $f(y) > f(x_0) = x_0$. This says the elements of the orbit of y , $\mathcal{O}_y$ are not moving towards x_0 . And since the only way

for an orbit to approach x_0 is from below, it must be the case that to converge to x_0 , the elements of $\mathcal{O}_y$ would have to jump to the other side of x_0 . But if x_0 is the largest fixed point for f , then $\mathcal{O}_y$ is monotonic and we are done with this case.



For an orbit of $y \in u(x_0)$, where $y > x_0$, to cross to the other side of x_0 , the graph of $f(x)$ must again cross ℓ and dip down sufficiently quickly (see the figure). Thus there exists a fixed point $y_0 > x_0$. Look at the structure of the graph of f . If the maximum of f on $[x_0, y_0]$ is y_0 , then $\mathcal{O}_y \rightarrow y_0$. Hence for it to be possible that $\mathcal{O}_y$ converge to x_0 , part of the interval $[x_0, y_0]$ must be mapped to an interval greater than y_0 (again, see the figure). This is actually possible, and will allow $c\mathcal{O}_y$ to converge to x_0 .

However, this is not a problem. Since the graph of $f(x)$ on $[x_0, y_0]$ lies completely ABOVE the $y = x$

line, the pre-image of y_0 will be in (x_0, y_0) , as an eventual fixed point. But so will the pre-image of that point, as well as its pre-image. In fact, if we go backwards, there will be a sequence of pre-images of y_0 , a backwards orbit which is monotonically DECREASING. Hence it must converge and must converge to the only fixed point, it can: x_0 . But that means precisely that as close as you like, there WILL BE a point $z_g \in U(x_0)$, an eventually fixed point, where $\mathcal{O}_{z_g}$ does NOT converge to x_0 .

Note that the other case, with all of $f(x)$ under x_0 near x_0 is similar. And I leave the case where $f'(x) = -1$ to you.

Exercise (EP6). I will prove this two ways. The book uses a claim, which I proved above. Let's prove this result first using the claim. Let $f(x) = \sqrt{x}$ for $x \in I$ defined in the claim above. Choose two points $x, y \in I$ and suppose without loss of generality that $y - x = t \geq 0$. Then $y = x + t$ and

$$d(x, y) = |x + t - x| = t.$$

For the images, we have $d(f(x), f(y)) = \sqrt{x+t} - \sqrt{x}$. By the claim above we know that $\sqrt{x+t} - \sqrt{x} \leq \frac{1}{2}t$ for $x \in I$. Thus

$$d(f(x), f(y)) = \sqrt{x+t} - \sqrt{x} \leq \frac{1}{2}t = \frac{1}{2}d(x, y).$$

hence f is a $\frac{1}{2}$ -contraction on I .

However, without all of this machinery, there is a more direct way to show this. Again, choose any two $x, y \in I$, where $x \neq y$. Then

$$\begin{aligned} d(f(x), f(y)) &= |\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right) \\ &= \left(\frac{1}{\sqrt{x} + \sqrt{y}} \right) |x - y| \leq \frac{1}{2} |x - y| \end{aligned}$$

since for all $x, y \geq 1$, the expression $\sqrt{x} + \sqrt{y} \geq 2$.

Exercise (EP8). The model for this function is given in the book: Example 2.2.4 and the function $\sqrt{x}$ on $[1, \infty)$. In our case, with $f(x) = \log(x-1)+5$ on $I = [2, \infty)$, we need a slight modification. Here, f is increasing on all of I , and differentiable, with $f'(x) = \frac{1}{x-1}$. However, by Proposition 2.2.3, we only know here then that $f(x)$ is 1-Lipschitz, since $f'(2) = 1$. Thus, we cannot say directly that f is a contraction. To cure this, notice that $f(2) = 5$ and indeed $f(I) = [5, \infty)$. Hence, after one iteration of f , we can regard f as a map on $[5, \infty)$. Here, since $f'(5) = \frac{1}{4}$, it is easy to see that f is a $\frac{1}{4}$ -contraction (on $[5, \infty)$ only, but remains a contraction throughout I in this case), and hence has a global attractor. This immediately rules out any periodic points of order higher than 1. To find the only fixed point, simply solve $x = f(x)$, or take any starting point and iterate. It is approximately 6.74903 using the natural log and 5.66925 using the common log.