

MATH 421 DYNAMICS

Week 9 Lecture 2 Notes

1. COUNTING PERIODIC ORBITS

It seems like we spend a lot of time in our study of dynamical systems on the classification and counting of periodic orbits of a map $f : X \rightarrow X$. To understand why, consider

- n -periodic points are the fixed points of the map f^n .
- Periodic points, like fixed points, have stability features.
- There are existence theorems for periodic points.
- many times, we can “solve” for them, without actually solving the dynamical systems.

Definition 1. For $f : X \rightarrow X$ a map, let

$$P_n(f) := \# \{x \in X \mid f^n(x) = x\}$$

be the set of all n -periodic points of f . And let

$$P(f) := \bigcup_{n \in \mathbb{N}} P_n(f).$$

Note that $P_n(f)$ also includes all m -periodic point when $m \mid n$. In particular, the 1-periodic points are the fixed points and these are in $P_n(f)$ for all $n \in \mathbb{N}$.

As a sequence, $\{P_n(f)\}_{n \in \mathbb{N}}$ can say a lot about f .

Recall $E_2 : S^1 \rightarrow S^1$, $E_2(z) = z^2$, $z = e^{2\pi i x} \in S^1$, or $E_2(s) = (2s \bmod 1)$, for $s \in S^1$, depending on your model for S^1 .

Proposition 2. $P_n(E_2) = 2^n - 1$, and all periodic points are dense in S^1 (i.e., $\overline{P(E_2)} = S^1$).

Proof. Using the model $E_2(z) = z^2$, we find that z is an n -periodic point if

$$\left(\dots \left((z^2)^2 \right) \dots \right)^2 = z \text{ or } z^{2^n} = z \text{ or } z^{2^n-1} = 1.$$

Thus every periodic point is an order- $(2^n - 1)$ root of unity (and vice versa). And there are exactly $2^n - 1$ of these, uniformly spaced around the circle. In fact, to any rational $\frac{p}{q} \in \mathbb{Q}$, the point $e^{2\pi i (\frac{p}{q})}$ is a q th root of unity. If $q = 2^n - 1$, for $n \in \mathbb{N}$, then $e^{2\pi i (\frac{p}{q})}$ is an order- n fixed point. Now as n goes to ∞ , the spacing between order- $(2^n - 1)$ roots of unity goes to 0. Hence any point $x \in S^1$ can be written as the limit of a sequence of these points. Hence will be in the closure of $P(E_2)$. $\square$

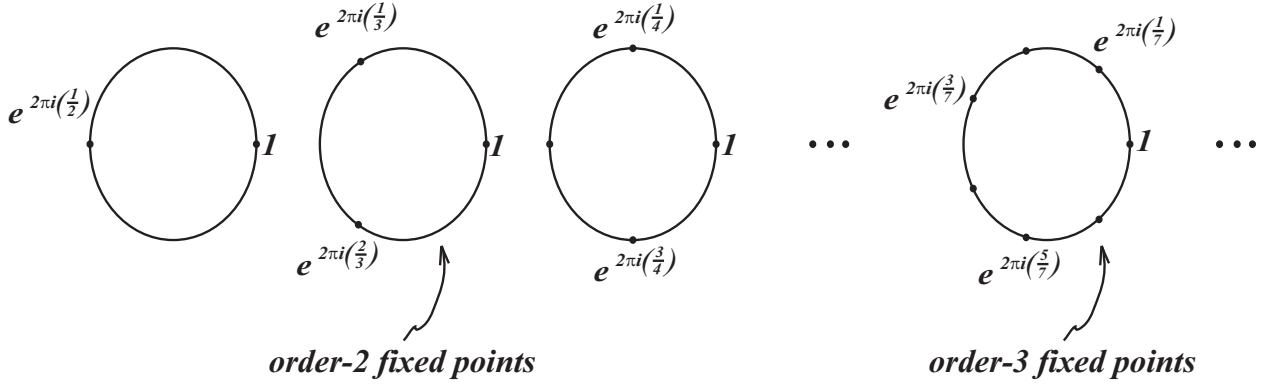


FIGURE 1. Order- n fixed points of E_2 spaced evenly in S^1

Example 3. z is 2-periodic if $z^{2^2-1} = z^3 = 1$. These are the points $z = e^{2\pi i(\frac{k}{3})}$, for $k = 1, 2$. To see how this works,

$$\begin{aligned} E_2(e^{2\pi i(\frac{1}{3})}) &= \left(e^{2\pi i(\frac{1}{3})}\right)^2 = e^{2\pi i(\frac{1}{3}) * 2} = e^{2\pi i(\frac{2}{3})}, \text{ while} \\ E_2(e^{2\pi i(\frac{2}{3})}) &= \left(e^{2\pi i(\frac{2}{3})}\right)^2 = e^{2\pi i(\frac{2}{3}) * 2} = e^{2\pi i(\frac{4}{3})} = e^{2\pi i(\frac{1}{3})}. \end{aligned}$$

We can calculate the growth rate of $P_n(E_2)$ in the obvious way: Define the *truncated* natural logarithm

$$\ln_+ x = \begin{cases} \ln x & x \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then define $p(f) = \overline{\lim}_{n \rightarrow \infty} \frac{\ln_+ P_n(f)}{n}$ as the relative logarithmic growth of the number of n -periodic points of f with respect to n .

For our case, then, where $E_2(z) = z^2$,

$$\begin{aligned} p(E_2) &= \overline{\lim}_{n \rightarrow \infty} \frac{\ln_+(2^n - 1)}{n} = \overline{\lim}_{n \rightarrow \infty} \frac{\ln_+ 2^n(1 - 2^{-n})}{n} \\ &= \overline{\lim}_{n \rightarrow \infty} \frac{\ln_+ 2^n + \ln_+(1 - 2^{-n})}{n} = \ln 2. \end{aligned}$$

This is the exponential growth rate of the periodic points of the map E_2 . Note that the growth factor is 2 at each stage, hence the exponential growth rate is the exponent of e which corresponds to the growth factor. Here $2 = e^{\ln 2}$.

Proposition 4. For $f : S^1 \rightarrow S^1$, $f(z) = z^m$, where $m \in \mathbb{Z}$ and $|m| > 1$,

$$P_n(f) = |m^n - 1|,$$

the set of all periodic points is dense in S^1 , and $p(f) = \ln |m|$.

Exercise 1. Show this for $m = -3$.

Here is an interesting fact: Let $f(z) = z^2$. The image of any small arc in S^1 is twice as long as the original arc. However, there are actually 2 disjoint pre-images of each small arc, and each is exactly half the size. Combined, the sum of the lengths of these two pre-images exactly matches the length of the image. Thus this expanding map on S^1 actually preserves length! Some notes about this:

- This is actually true for all of the expanding maps $E_m : S^1 \rightarrow S^1$, $E_m(z) = z^m$, where $m \in \mathbb{Z}$, and $|m| > 1$.
- This is a somewhat broadening of the idea of area preservation for a map. When the map is onto but not 1-1 (in this case, the map is 2-1), the relationship between pre-image and image is more intricate, and care is needed to understand the relationship well.

1.1. The quadratic family. For $\lambda \in \mathbb{R}$, let $f_\lambda : \mathbb{R} \rightarrow \mathbb{R}$, where $f_\lambda(x) = \lambda x(1 - x)$ is also called the logistic map on $\mathbb{R}$. For $\lambda \in [0, 4]$, we can restrict to $I = [0, 1]$, and $f_\lambda : I \rightarrow I$ is the interval map family we partially studied already. In fact, we can summarize our results so far: For $\lambda \in [0, 3]$, the dynamics are quite simple. There are only fixed points, and no nontrivial periodic points, and all other points are asymptotic to them. The fixed points are at $x = 0$ and $x = 1 - \frac{1}{\lambda}$.

Some new facts:

- (1) for $\lambda \in [3, 4]$, a LOT happens! (we will get to this later in the course.)
- (2) for $\lambda > 4$, I is not invariant.
- (3) since f_λ is quadratic, f_λ^n is at most of degree 2^n . Thus the set of n -periodic points must be solutions to the equation $f_\lambda^n(x) = x$. Bringing x to the other side of the equation, the set $P_n(f_\lambda)$ must consist of the roots of an (at most) 2^n -degree polynomial. Hence

$$P_n(f_\lambda) \leq 2^n, \text{ for all } \lambda \in \mathbb{R}.$$

- (4) For $\lambda > 4$, many points escape the interval I . However, as we will see, many points have orbits which do not. We can still talk about the map on the set of all of these points....

Let $\lambda > 4$, and consider the first iterate of f_λ . Notice (see the figure), that the intervals I_1 and I_2 are both mapped onto $[0, 1]$ and that each contains exactly one fixed point. Under the second iterate of the map, f_λ^2 , only points in the 4 intervals J_i , $i = 1, 2, 3, 4$ remain in $[0, 1]$. Here there are 4 fixed points (again one in each interval). But notice that only two of them are new, y_1 and y_2 . These two new ones are period-2 points that are not fixed points. See in the cobwebbed figure the period-2 orbit on the right of the figure.

Continue iterating in this fashion, and one can see that there will be

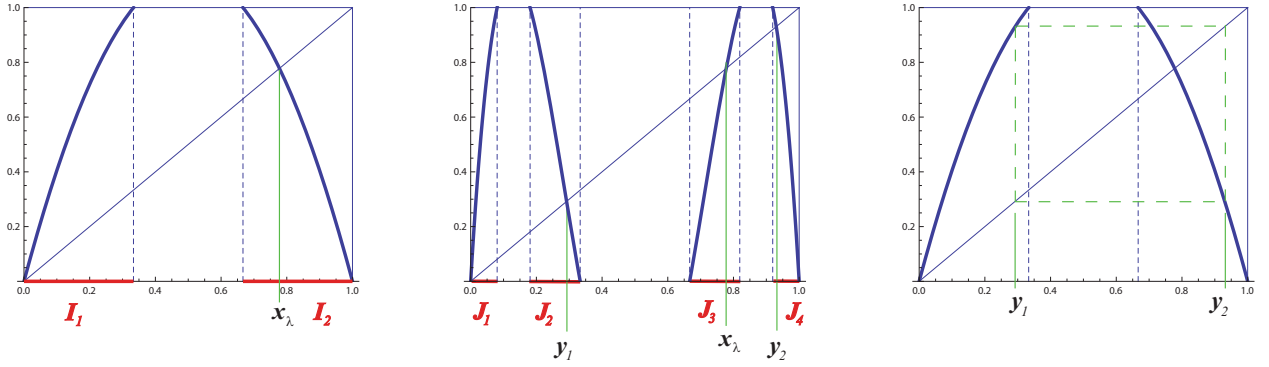


FIGURE 2. The map f_λ , f_λ^2 , and the period-2 orbit

- 2^n intervals of points that remain in $[0, 1]$ after n -iterates.
- The next iterate of f_λ maps each of these 2^n intervals onto $[0, 1]$, creating a single fixed point in each interval (of f_λ^n).
- You can see then (and one can prove this by induction) that $P_n(f_\lambda) = 2^n$, when $\lambda > 4$.

This quadratic family is an example of a *unimodal* map: A continuous map defined on an interval that is increasing to the left of an interior point and decreasing thereafter.

Proposition 5. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous with $f(0) = f(1) = 0$ and suppose there exists $c \in (0, 1)$ such that $f(c) > 1$. Then $P_n(f) \geq 2^n$. If, in addition, f is unimodal and expanding, then $P_n(f) = 2^n$.*

Definition 6. A map $f : [0, 1] \rightarrow [0, 1]$ is *expanding* if

$$|f(x) - f(y)| > |x - y|$$

on each interval of $f^{-1}([0, 1])$.

Examples of expanding maps include the logistic map for suitable values of $\lambda > 4$, and the circle maps E_m , where $m \in \mathbb{Z}$ and $|m| > 1$ (you should modify the definition here to include maps of S^1). Note here:

- In the Proposition, the condition $f(0) = f(1) = 0$ and continuity ensure that the map will “fold” the image over the domain, and
- the condition $f(c) > 1$ ensures the folding will be complicated, with lots of points escaping, while lots of points will not.

1.2. Expanding Maps. Here is a better definition of an expanding map (albeit limited now to circle maps):

Definition 7. A C^1 -map $f : S^1 \rightarrow S^1$ is expanding if $|f'(x)| > 1$, $\forall x \in S^1$.

Example 8. It should be obvious by this definition that the map E_m , where $m \in \mathbb{Z}$ and $|m| > 1$ is expanding, since $E_m(x) = mx \pmod{1}$ is differentiable and $|E'_m(x)| = |m| > 1$ for all $x \in S^1$.

Recall that the degree of a circle map is a well defined property that measures how many times the image of a map of S^1 winds itself around S^1 .

Lemma 9. *Let $f, g : S^1 \rightarrow S^1$ be continuous. Then*

$$\deg(g \circ f) = \deg(g)\deg(f).$$

Proof. Degree is defined via a choice of lift: Given lifts $F, G : \mathbb{R} \rightarrow \mathbb{R}$ of these two maps, we have for $s \in S^1$ and $k \in \mathbb{Z}$,

$$G(s+k) = G(s+k-1) + \deg(g) = G(s+k-2) + 2\deg(g) = \cdots = G(s) + k\deg(g).$$

But this means

$$G(F(s+1)) = G(F(s) + \deg(f)) = G(F(s)) + \deg(f)\deg(g).$$

□

Example 10. $\deg(f^n) = (\deg(f))^n$.

Hence we can use this to show:

Proposition 11. *If $f : S^1 \rightarrow S^1$ is expanding, then $|\deg(f)| > 1$ and $P_n(f) = |(\deg(f))^n - 1|$.*

Here is a 2-dimensional version of periodic point growth. Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $L(x, y) = (2x + y, x + y)$. We can also write L as the linear vector map

$$L(\mathbf{x}) = A\mathbf{x}, \text{ where } A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

We know that since A has integer entries, it takes integer vectors to integer vectors, and hence descends to a map on the two torus $\mathbb{T}^2$. Indeed, if $\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ satisfy $\mathbf{x}_1 - \mathbf{x}_2 \in \mathbb{Z}^2$, then

$$L(\mathbf{x}_1 - \mathbf{x}_2) = L(\mathbf{x}_1) - L(\mathbf{x}_2) \in \mathbb{Z}^2.$$

But then $L(\mathbf{x}_1) - L(\mathbf{x}_2) = \mathbf{0} \pmod{1}$, which means $L(\mathbf{x}_1) = L(\mathbf{x}_2) \pmod{1}$. Hence the map L induces a map on $\mathbb{T}^2$ which assigns

$$(x, y) \mapsto (2x + y \pmod{1}, x + y \pmod{1}).$$

We will call this new *induced map* on the torus $F_L : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, where

$$F_L(\mathbf{x}) = A\mathbf{x}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{x} \in \mathbb{T}^2.$$

Some notes:

- This map is an *automorphism* of $\mathbb{T}^2$: A homeomorphism that preserves also the ability of points on the torus to be added together (multiplied, if one defines the multiplication correctly).
- F_L is also invertible since it is an integer matrix of determinant 1. The inverse map $F_L^{-1} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is given by the matrix $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.
- The eigenvalues of F_L (really the eigenvalues of A) are the solutions to the quadratic equation $\lambda^2 - 3\lambda + 1 = 0$, or

$$\lambda = \frac{3 \pm \sqrt{5}}{2}.$$

Note that

$$\lambda_1 = \frac{3 + \sqrt{5}}{2} > 1, \quad \text{and} \quad \lambda_2 = \frac{3 - \sqrt{5}}{2} < 1,$$

so that the matrix defining F_L is a hyperbolic matrix (determinant-1 with eigenvalues off the unit circle in $\mathbb{C}$).

Question 12. *How does F_L act on $\mathbb{T}^2$?*

Really the answer to this question relies on how L acts on $\mathbb{R}^2$. Watching the model of $\mathbb{T}^2$ as the unit square in $\mathbb{R}^2$ as it is acted on by L provides the means to study the F_L action on $\mathbb{T}^2$. This is the two dimensional version of studying a lift of a circle map on $\mathbb{R}$ as a means of studying the circle map.

Linear maps of the plane take lines to lines. Hence they take polygons to polygons, and, in this case, they take parallelograms to parallelograms. The image of the unit square can be found by simply finding the images of the four corners of the square and constructing the parallelogram determined by those points. In this case, we have the figure. But there is more. L is area preserving. Hence the image of the parallelogram will also have area 1. And due to the equivalence relation given by the exponential map on $\mathbb{R}^2$, every point in the image of the unit square has a representative within its equivalence class INSIDE the unit square. We can reconstruct the unit square by translating back all of these outside points back into the square. This becomes the image of points on the torus back into the torus.

See the drawing, where

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$