MATH 421 DYNAMICS

Week 8 Notes

1. Newtonian Systems of Classical Mechanics

Your previous work in ODEs suggested a general premise about systems of differential equations. If they are defined “nicely”, then the present state of a mechanical system determines its future evolution through other states uniquely. One can place this in the language of dynamical systems to say that if a mathematical construction accurately models a mechanical system, than the construction determines a dynamical system on the space of all possible states of the system. The trick in many cases is to well understand what constitutes a state of a mechanical system. To start, given a mechanical system, the *configuration space* of the system is the set of all possible positions (value combinations of all of its variables) of the system. The *state space*, rather, is the set of all possible states the system can be in. This is usually much broader a description.

For example, consider the pendulum, a mass is attached to the free end of a massless rigid rod, while the other end of the rod is fixed. The set of all possible configurations of the pendulum is simply a copy of S^1. However, for each configuration, the pendulum is in a different state depending on what the mass' velocity is when it resides in a configuration. One can think of all possible states as the space $S^1 \times \mathbb{R}$. This reflects the data necessary to completely determine the future evolution of the pendulum by a knowledge of its position and velocity at a single moment, and the evolution equation which is a second-order, possibly non-linear and non-autonomous, ODE in the general form

$$\ddot{x} = f(t, x, \dot{x}).$$

In the case of a pendulum, time is not explicit on the right hand side, and the equation is autonomous. Under the standard practice of converting this ODE into a system of two first order ODEs, we can interpret the evolution as giving a vector field on the state space $S^1 \times \mathbb{R}$, with coordinates x and $\dot{x}$. This vector field determines a flow, which solves the ODE and determines the future evolution of the system based on knowledge of the state of the system at a particular moment in time.

Many systems behave in a way that their future states are completely determined by their present position and velocity, along with a notion of how they are changing. In classical (Newtonian) mechanics, Newton's Second Law of motion states roughly that the force acting on an object is proportional to how the velocity of the object is changing. The is the famous equation $f = ma$, where f is the total force acting on the object and a is its acceleration. As the velocity depends on the current position of an object, a good notion of how an object moves through a space under the influence of a force is completely determined by how its

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position and velocity are changing, at least when the force is static:

$$f(x) = ma = m\ddot{x} = m\frac{d^2x}{dt^2}.$$

This is a special case of the general second order ODE mentioned above.

Example 1. An object under the influence of only gravity satisfies Newton's Second Law and the differential equation is $\ddot{x} = -g$, where g is the gravitational constant. This is solved by integrating the "pure time" ODE twice

$$x(t) = -g\frac{t^2}{2} + v_0 t + s_0,$$

where v_0 and s_0 are the initial velocity and initial position, respectively (the two constants of the integrations).

Example 2. Harmonic Oscillator. Recall Hooke's Law: the amount an object is deformed is linearly related to the force causing the deformation. This translates to $\ddot{x} = -kx$, which is autonomous. Solutions are given by

$$x(t) = a\sin\sqrt{k}t + b\cos\sqrt{k}t,$$

where a and b are related to the initial starting position and velocity of the mass.

As stated above, note that any ODE of the form $\ddot{x} = f(t, x, \dot{x})$ can be converted to a system of two first order (generally coupled) ODEs of the form

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= f(t, x, v) \end{aligned}$$

which defines a vector field (a static vector field if t does not appear explicitly in the equations) on the (x, v)-state space. For Newton's Equation, the equivalent system is $\dot{x} = v$ and $\dot{v} = \frac{1}{m}f(x)$.

Note: For the n-system governed by Newton's Law $f = ma$, we get the $2n$-system of first order equations defined as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{f}(t, \mathbf{x}, \mathbf{v}) \end{aligned}$$

The state space consists of the $2n$-dimensional vectors $\begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}$. The vector field of this $2n$-system attaches the vector $V = \begin{bmatrix} \mathbf{v} \\ \frac{1}{m}\mathbf{f}(\mathbf{x}) \end{bmatrix}$ to each point $\begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}$. The divergence of this vector field is

$$\nabla \cdot V = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(v_i) + \sum_{i=1}^{n} \frac{\partial}{\partial v_i}\left(\frac{1}{m} f_i(\mathbf{x})\right) = 0.$$

Hence the flow preserves volume.

Remark 3. This fact is now true for general Newtonian systems. One facilitating idea in Newtonian physics is to, in essence, factor out the mass. Define a new variable (a coordinate) $q := mx$, and then switch from the velocity coordinate to $p := mv$ as the (linear) momentum. Then $\dot{q} = m\dot{x} = mv$, and $f = ma = f = m\dot{v} = \dot{p}$, and the system becomes $\dot{q} = p$ and $\dot{p} = f\left(\frac{1}{m}q\right)$. Not only does this make the system easier to work with, it exposes some hidden symmetries within the equations of conservative systems. This forms the basic framework for what are called Hamiltonian dynamics.

Now assume that the force $f(x)$ is a gradient field (this means that the force is the gradient of a function of position alone, or $f = -\nabla V$ for some $V(x)$. Then

$$f(x) = ma = m\dot{v} = -\nabla V.$$

Here, the function V is called the potential energy (energy of position), and the energy of motion, the kinetic energy is

$$K = \frac{1}{2}m\left|\left|v\right|\right|^2 = \frac{1}{2}m(v \cdot v).$$

The total energy $H = K + V$ satisfies

$$\frac{d}{dt}(H) = \frac{dK}{dt} + \frac{dV}{dt} = m\dot{v} \cdot v + \sum_{i=1} n\frac{\partial V}{\partial x_i} \cdot \frac{\partial x_i}{\partial t} = m\dot{v} \cdot v + \nabla V \cdot v = \left(\nabla V + m\dot{v}, v\right) = 0.$$

The conclusion is the total energy H is conserved as one evolves in a system like this. As H is a function defined on the state space given by the vectors x and mv, the solutions to the system of ODEs are confined to the level sets of this function. A system like this is called *conservative*, and is characterized by the idea that the force field is a gradient field. You have seen this before in a different guise:

1.1. **Exact Differential Equations.** Consider the nonlinear system of 2 first-order, linear, autonomous differential equations in 2 variables

$$\begin{aligned} \dot{x} &= 4 - 2y \\ \dot{y} &= 12 - 3x^2. \end{aligned} \tag{1}$$

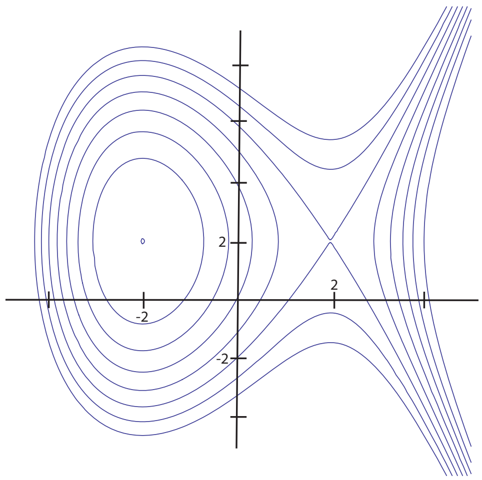
This system can also be written by the single differential equation

$$(12 - 3x^2)dx - (4 - 2y)dy = 0. \tag{2}$$

Note that this equation is exact, and separable, and upon integration, one obtains

$$4y - y^2 = 12x - x^3 + C.$$

this defines our solutions implicitly. In fact, we can use this directly.



Define a function $H(x,y) = 4y - y^2 - 12x + x^3$. Then H is conserved by the flow, and the flow must live along the constant level sets of H (the sets that satisfy $H(x,y) = C$.) These sets are given by the figure.

Now recall in Section **??** that an ODE $M\,dx + N\,dy = 0$ is exact if $M_y = N_x$ (this notation again refers to the first partial derivatives of the functions with respect to the subscripts). The reason is vector-calculus in nature: The solution is a function $\varphi(x,y) = C$ satisfying $\frac{\partial\varphi}{\partial x} = M$ and $\frac{\partial\varphi}{\partial y} = N$. The condition for exactness is simply the statement that for any C^2 function $\varphi(x,y)$, the mixed partials are equal:

$$M_y = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial\varphi}{\partial x} = \frac{\partial^2\varphi}{\partial y\partial x} = \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial}{\partial x}\frac{\partial\varphi}{\partial y} = \frac{\partial N}{\partial x} = N_x.$$

But we can now go even further: The vector field $\mathbf{F}(x,y) = (4-2y, 12-3x^2)$ of the system in Equation 1 corresponds to the exact ODE given by Equation 2 when $M = 12 - 3x^2$ and $N = -(4-2y)$, or $\mathbf{F}(x,y) = (-N, M)$. With this,

$$div(\mathbf{F}) = \frac{\partial}{\partial x}(-N) + \frac{\partial}{\partial y}(M) = -N_x + M_y = 0.$$

The vector field $\mathbf{F}$ is conservative, and the flow will preserve volume (area) in the plane. This is a general fact for vector fields of exact ODEs and leads directly to:

Proposition 4. *The flow of an exact ODE in $\mathbb{R}^2$ preserves volume in phase space.*

Corollary 5. *Equilibria of exact ODEs in $\mathbb{R}^2$ can only be saddles or centers.*

The repercussions of these facts are quite important: For instance, $\mathbf{p} = (-2,2)$ is an equilibrium solution of Equation **??**. What is its type and stability (forgetting the figure for a moment, that is) of $\mathbf{p}$? We can linearize this Almost Linear System (See Section **??**) at $\mathbf{p}$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{\partial(-N)}{\partial x}(-2,2) & \frac{\partial(-N)}{\partial y}(-2,2) \\ \frac{\partial M}{\partial x}(-2,2) & \frac{\partial M}{\partial y}(-2,2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 12 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

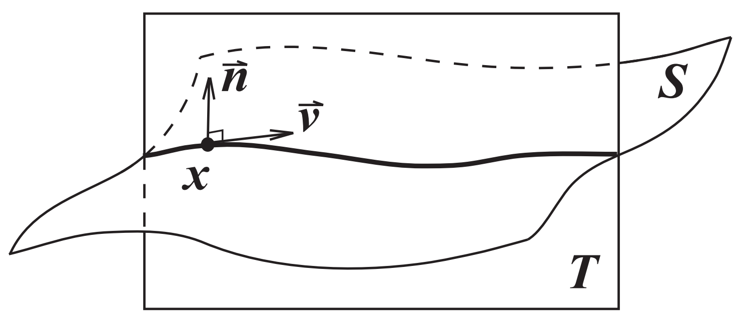
The eigenvalues $r = \pm\sqrt{24}$ ar epurely imaginary. Hence teh linearlized equilibrium at the origin is a center. But centers are NOT *structurally* stable, in that a small perturbation in a center may result in a sink or a source, as well as a center (the eigenvalues may take on small real parts, either negative or positive). Hence we cannot by itself declare that $\mathbf{p}$ is in fact a center via the linearized system.

However, with the *additional* knowledge that **F** is conservative, then sinks and sources are not possible, and in fact, the point **p**, an equilibrium of the nonlinear system, MUST be a center. Such is an import of phase volume preservation.

Exercise 1. Complete the phase diagram for this ODE by noting directions of motion along the level curves. Also, note the values of the level sets corresponding to the two equilibria solutions. Finally, show analytically that the equilibrium at $(2, 2)$ is unstable, while the equilibrium at $(-2, 2)$ is stable.

1.2. **Newton's First Law.** An object not under the influence of an external force will move linearly and with constant (maybe zero) velocity. How does this notion of linear (constant velocity) motion appear

- in Euclidean space?
- on $\mathbb{T}^n$?
- On S^2?
- On an arbitrary metric space? Here we must get a better understanding of just what a straight line is in a possibly curved space. We can use the metric to define a straight line as the path that is the shortest distance between two points. This path is called a *geodesic*.



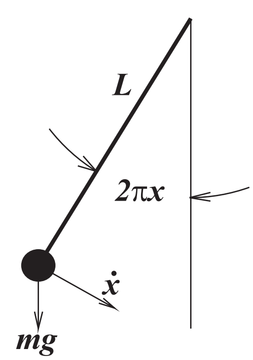
On a smooth surface $S \subset \mathbb{R}^3$, the Euclidean metric on $\mathbb{R}^3$ induces a metric on S. Choose a point $x \in S$. The surface has a well-defined tangent plane to S at x. With this tangent plane, we can choose a normal **n** to the surface at x, as well as a desired direction **v** in the tangent plane. Now, for a particle moving freely along the surface S in the direction of **v** at x, the ONLY force acting on the particle is the force keeping it on S. Thus, the acceleration vector of the particle is in the direction of **n**. With no component of the force in the direction of motion, the speed $||\mathbf{v}||$ is constant along this intersection line.

Question 6. *What do the geodesics look like on S^2?*

1.3. **The Planar Pendulum.** One can model the planar pendulum by the autonomous second order differential equation

$$2\pi m L\ddot{x} + mg\sin(2\pi x) = 0. \tag{3}$$

Some notes:



- This is the undamped pendulum as stated. If one were to consider damping, one can model this by adding a term involving $\dot{x}$. A common one is $cL\dot{x}$.
- This equation can be rewritten as
$$\frac{2\pi L}{g}\ddot{x} + \sin(2\pi x) = 0.$$
- To simplify even further, one can scale time by $\tau = \frac{t}{T}$, where $T = \sqrt{\frac{g}{2\pi L}}$. Then we get
$$\ddot{x} + \sin(2\pi x) = 0.$$

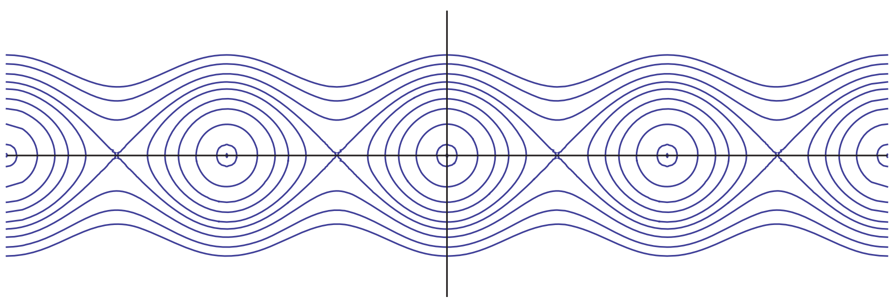
So the model becomes

$$\begin{aligned}\dot{x} &= v\\ \dot{v} &= -\sin 2\pi x,\end{aligned}$$

which is Newtonian with $f(x) = -\sin 2\pi x$. Here the kinetic energy is $K = \frac{1}{2}v^2$, and V is the potential energy, where

$$f(x) = -\nabla V, \qquad \text{and} \qquad V = \int \sin 2\pi x\, dx = -\frac{1}{2\pi}\cos 2\pi x.$$

The total energy is $H = K + V = \frac{1}{2}v^2 - \frac{1}{2\pi}\cos 2\pi x$ and is conserved. Hence motion is along the level sets of H.



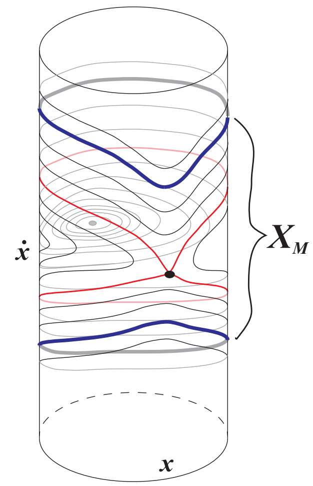
Some dynamical notes:

- For low energy values $H \in \left(-\frac{1}{2\pi}, \frac{1}{2\pi}\right)$, motion is periodic and all orbits are closed.
- For high energy values $H > \frac{1}{2\pi}$, motion looks unbounded. But is it really? What is the pendulum actually doing for these energy values? Recall the idea from before that the actual phase space is a cylinder (the horizontal coordinate – the position of the pendulum – is angular and hence periodic). Hence motion is still periodic and orbits are still closed.

- What happens for the energy value $H = \dfrac{1}{2\pi}$? What is the lowest energy value? What are the energy values of the equilibria solutions?

Back to Poincare Recurrence. This system is conservative and hence exhibits phase volume preservation (incompressibility). What can we say about the recurrent points? Theorem **??** required a finite volume domain to establish the density of recurrent points. On the phase cylinder, we can create a finite volume domain simply by bounding the total energy $H < M$, for some $M > -\frac{1}{2\pi}$, so

$$X_M = \left\{ (x,v) \in S^1 \times \mathbb{R} \;\middle|\; H(x,v) < M \right\}.$$



Now by Theorem **??**, almost all points on X_M are recurrent. Can you find points that are not recurrent in the phase space? Can you classify them? Look for points $\mathbf{x}$, where $\mathbf{x} \notin \omega(\mathbf{x})$. That the phase plane for the pendulum is actually a cylinder is an extremely important concept, if not for this reason alone.

Next class, we will continue with a few examples of such systems before we analyze the structure these systems have.