

MATH 421 DYNAMICS

Week 7 Lecture 2 Notes

1. CONSERVATIVE SYSTEMS: SYSTEMIC RECURRENCE

In Chapter 4, we first looked at what was considered “recurrent” behavior at a point in a dynamical system, which roughly means that the orbit of a point passes arbitrarily close to the point. This worked well in the classification of circle rotations, since either the orbit of a point was closed (the orbit was periodic; the rotation was rational) or the orbit was dense (for an irrational rotation). In either case, every point was recurrent. The same was true for the linear toral flows and their time- t maps.

Contrast this with the dynamical systems that we studied in Chapters 2 and 3. Here, with examples like contracting maps and sinks and sources, the only recurrent points were the fixed and periodic points, and there were very few of those in each system. More generally, maps can exhibit much more complicated behavior. To understand this behavior, we will have to broaden our idea of how to study such systems. This chapter begins this study.

To start, let’s change our perspective. Given a dynamical system, let’s not worry about how an individual orbit behaves so much as how whole families of nearby orbits evolve. This would be more like following all of the orbits that start in a small open subset of the state space over the evolution of the map. For a contraction, this would be easy and not very insightful. (why?) But for a general map, this idea can be quite interesting.

2. INCOMPRESSIBILITY

The notion of incompressibility in a dynamical system means that positive volume domains in the state space do not change their volume as the orbits of their points evolve. This notion is also called *phase volume preservation*. Suppose we have a dynamical system where this property holds; as one evolves via a flow, or iterates via a map, the volume of a small domain does not change. Then the volume is said to be preserved by the flow (respectively, map), or the volume is invariant under the flow (respectively, map). Obvious examples include linear flows in $\mathbb{R}^n$, rotation maps on S^1 (remember that volume in a space like $\mathbb{R}$ or S^1 is just length, and in dimension 2 is just area), and linear toral flows. Examples which do not preserve volume include contraction maps, and flows (defined by ODEs) that include sinks and sources (saddles and centers, maybe, though).

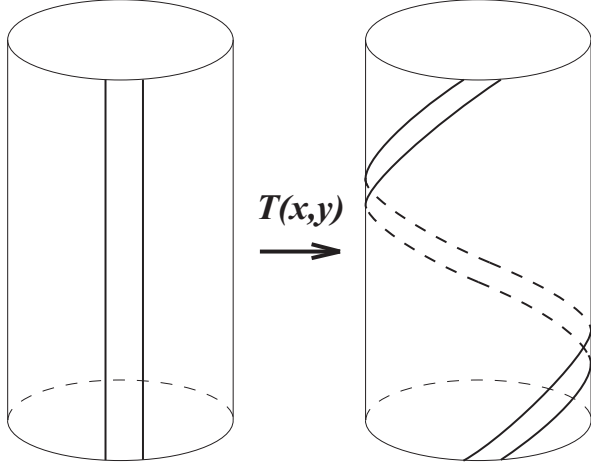
In fact, if the map is by isometries, or the flow has all of its time- t map given by isometries, then the volume will be preserved. This should be obvious, as if all of the distances between the points of a small domain are preserved, the volume cannot change. The converse is not true however. Lots of maps and flows preserve volume but are not isometries.

Example 1. Consider the linear twist map on the cylinder

$$T : S^1 \times [0, 1] \longrightarrow S^1 \times [0, 1], \quad T(x, y) = (x + y, y).$$

What does this twist look like? See the figure below.

Exercise 1. Show that (1) T is not an isometry, but (2) T preserves area on the cylinder.



Now, let's consider a linear map on $\mathbb{R}^n$:

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad f(\mathbf{x}) = A\mathbf{x},$$

where A is a $n \times n$ matrix. Choose an orthonormal basis for $\mathbb{R}^n$. Then the standard cube C whose sides are the basis vectors (this is the “unit cube” relative to the basis) will be mapped by f to a parallelepiped. What would be the volume of this image? Well, here

$$\text{vol}(f(C)) = |\det A|.$$

What would be the conclusion one can draw from this? This is simply the infinitesimal

version of any smooth map on $\mathbb{R}^n$. Indeed, let's start with a better idea of what kind of sets have positive volume in $\mathbb{R}^n$. Recall in any metric space X , we can define a small open set via an inequality:

$$B_\epsilon(x) = \left\{ y \in X \mid d(x, y) < \epsilon \right\}.$$

Definition 2. A subset $U \in X$ is called *open* if $\forall x \in U, \exists \epsilon > 0$ such that $B_\epsilon(x) \in U$. A subset is called *closed* if its complement is open.

Definition 3. A *domain* in X is either an open subset of X , or the closure of an open subset of X .

This last definition ensures that a domain has non-zero volume, although the volume may be infinite. In $\mathbb{R}^n$ with the standard Euclidean metric, the ϵ -balls have volume $\frac{4}{3}\pi\epsilon^3 > 0$, when $\epsilon > 0$.

Proposition 4. *let $U \in \mathbb{R}^n$ be an open domain. A differentiable map $f : U \rightarrow \mathbb{R}^n$ preserves volume iff $|\det(Df_x)| = 1, \forall x \in U$.*

The *Jacobian matrix* of a function like f is the matrix of partial derivatives of f , and their values at a point $x \in U$ become the derivative matrix at that point Df_x . It is common to refer to the determinant of this matrix the *Jacobian* of f , $Jac(f)$.

Definition 5. A map $f : U \rightarrow \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is a domain, preserves orientation if $\forall x \in U$, $Jac(f) > 0$.

“Nice” ODEs (where solutions exist and are unique everywhere, for example), are always orientation preserving. Recall the relationship between the time-1 map of any linear ODE system on $\mathbb{R}^2$. It always has eigenvalues which were related to those of the original flow by the exponential map. Under the exponential map, the time-1 map will always have a positive Jacobian (why?).

More generally, let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be an ODE on $\mathbb{R}^n$. Then the function $\mathbf{f}$ defines a vector field on $\mathbb{R}^n$ (to each point $\mathbf{x}$ we attach the vector $\mathbf{f}(\mathbf{x})$). Remember this fact about the divergence of the vector field from Calculus III?

Proposition 6. If the divergence of the vector field $\mathbf{f}$ vanishes (that is, if $\text{div}(\mathbf{f}) = 0$), then $\mathbf{f}$ preserves volume.

Theorem 7. let X be a finite volume domain in $\mathbb{R}^n$ or $\mathbb{T}^n$, and $f : X \rightarrow X$ be an invertible, volume preserving C^1 -map. Then $\forall x \in X$ and $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$ such that

$$f^n(B_\epsilon(x)) \cap B_\epsilon(x) \neq \emptyset.$$

Proof. This can be easily see as follows: Suppose $\exists x \in X$, and $\exists \epsilon > 0$ such that $\forall n \in \mathbb{N}$

$$f^n(B_\epsilon(x)) \cap B_\epsilon(x) = \emptyset.$$

Since f is volume preserving, we must have at the n th iterate:

$$\infty > \text{vol}(X) > \sum_{i=1}^n \text{vol}(f^i(B_\epsilon(x))) = n \cdot \text{vol}(B_\epsilon(x)).$$

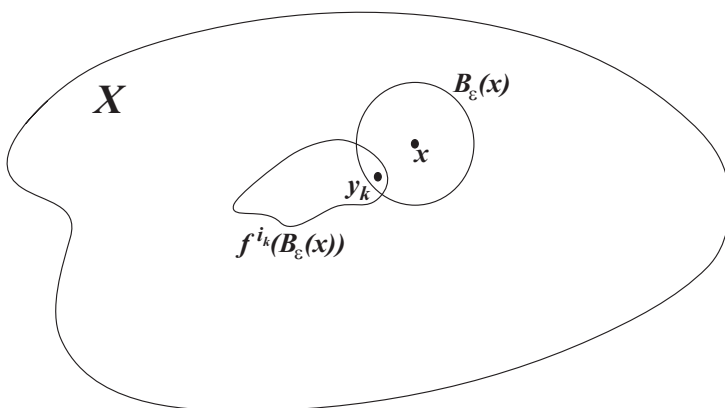
But for all choice of $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} n \cdot \text{vol}(B_\epsilon(x)) = (\text{vol}(B_\epsilon(x))) \lim_{n \rightarrow \infty} n = \infty$$

since $\text{vol}(B_\epsilon(x)) > 0$. This contradiction establishes the proof. $\square$

This gives us an immediate consequence:

Corollary 8. For $f : X \rightarrow X$ as above, $\forall x \in X$, there exists a sequence $\{y_k\} \rightarrow x$ and a sequence $\{i_k\} \rightarrow \infty$ where $\{f^{i_k}(y_k)\} \rightarrow x$.



See the figure to get an idea of what is going on. Given any small neighborhood of $B_\epsilon(x) \subset X$, There will be a iterate (here, the i_k -th iterate) of f in the forward orbit of $B_\epsilon(x)$ which will intersect $B_\epsilon(x)$.

Choose any point y_k in the intersection. Now choose a new $\epsilon > 0$ where $\epsilon < d(x, y_k)$, and repeat the procedure. Play this game for a decreasing sequence of ϵ 's going to 0. At each stage, you produce a y_k close

to x that has a forward iterate that is even closer. In the limit, you show that arbitrarily close to x is a recurrent point. As the choice of x does not matter, you have that recurrent points are almost everywhere.

Exercise 2. Produce this sequence.

Recall Definition ?? on a point being recurrent. We can extend that notion now to

Definition 9. For $f : X \rightarrow X$ a continuous map of a metric space, a point $x \in X$ is called

- *positively recurrent* with respect to f if $\exists$ a sequence $\{n_k\} \rightarrow \infty$ such that $\{f^{n_k}(x)\} \rightarrow x$,
- if f is invertible, *negatively recurrent* if $\exists$ a sequence $\{n_k\} \rightarrow -\infty$ such that $\{f^{n_k}(x)\} \rightarrow x$,
- *recurrent* if it is both positively and negatively recurrent.

Definition 10. For $f : X \rightarrow X$ a continuous map of a metric space, the set

$$\omega(x) = \overline{\bigcap_{n \in \mathbb{N}} \left\{ f^i(x) \mid i \geq n \right\}}$$

is the set of all accumulation points of the orbit of x . It is called the ω -limit set of $x \in X$ with respect to f . For f an invertible map on X , the set

$$\alpha(x) = \overline{\bigcap_{n \in \mathbb{N}} \left\{ f^{-i}(x) \mid i \leq n \right\}}$$

is called the α -limit set of x with respect to f .

Note: $x \in X$ is positively recurrent if $x \in \omega(x)$ (if x is in its own ω -limit set).

Exercise 3. Show, by construction, that $\forall \alpha \in \mathbb{R}$, all points of S^1 are recurrent under the rotation map R_α .

Exercise 4. Show the same by construction for a translation on $\mathbb{T}^2$.

Theorem 11. Let X be a closed finite-volume domain in $\mathbb{R}^n$ or $\mathbb{T}^n$ and $f : X \rightarrow X$ an invertible volume preserving map. Then the set of recurrent points for f is dense in X .

Note; This does not mean that all points are recurrent, not that there may be tons of points whose ω -limit sets do not include the original point. It does mean that every point either

is recurrent, or has a recurrent point arbitrarily close to it. We won't prove this here. The proof is in the book on page 160. Instead, let's skip ahead to Section 6.2.