MATH 421 DYNAMICS

Week 7 Lecture 1 Notes

0.1. **Application: The Kepler Problem.** One more application of linear toral flows: The Kepler Problem: Consider two point masses moving in an inverse square gravitational field. Assume that they do not interact or influence each other. Then their equations of motion are second order homogeneous (and separable). Total energy is conserved (this is actually the solution function when using the method one is taught in ODEs to solve separable equations) as is total angular momentum. Hence flow is planar, and confined to an ellipse (hence each point mass has periodic flow of some period independent of the other). For two particles, the flow would be confined to a torus again (two separate periodic flows, although in this case, the flow would look a bit more like that of the flow in $\mathbb{R}^4$ above. The flow would be linear in a space where momenta is used instead of velocity. Then, one can easily say whether the two point mass system will ever reach its starting positions simultaneously again based on whether the ratio of the momenta is rational or not. Pretty easy result for such a complicated system. Two other thoughts:

- **Q.** Can you now see the similarity between the Kepler Problem and the HW assignment I gave you concerning the rotation of the earth and the lunar rotation?
- **Q.** What would the Kepler Problem look like for three point masses in the same field? Where would the resulting flow reside? Can we make a concluding statement about the flow in such an easy way as for only two point masses?

We will return to the last questions in short order. But first, let's return to circle maps. At this point, however, let's talk about more general complicated maps on such a simple space as S^1.

1. Invertible S^1-maps.

Let's return to maps on the circle, and try to gain more general information than by using simply rigid rotations. Again, think of S^1 as the identification space $S^1 = \mathbb{R}/\mathbb{Z}$, given by the level sets of the map

$$\pi : \mathbb{R} \to S^1, \quad \pi(x) = [x].$$

One easy way to think about $[x]$ is to simply take any real number and disregard the integer part. Thus $[2.13] = .13$, and $[e] = e - 2$. We will call π a projection of $\mathbb{R}$ onto S^1

Proposition 1. *For any continuous map $f : S^1 \to S^1$, there exist an associated map $F : \mathbb{R} \to \mathbb{R}$, called a* lift *of f to $\mathbb{R}$, where*

$$f \circ \pi = \pi \circ F, \qquad \textit{equivalently } f([x]) = [F(x)].$$

Some Notes:

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- One way to see this is vis the commutative diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

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- The lift F is unique up to an additive constant (sort of like how the anti-derivative of a function is unique only up to an additive constant, right?)
- The quantity

$$\deg(f) = F(x+1) - F(x)$$

is well-defined for all $x \in \mathbb{R}$ and is called the *degree* of f.
- If f is a homeomorphism, then $|\deg(f)| = 1$.
- The structure of F is quite special. It looks like the sum of a periodic function with the line $y = (\deg(f))\, x$. This is due to the structure of the projection π.

So just how much information about f can we learn by the study of the lifts of f? Certainly, maps on $\mathbb{R}$ are fairly easy to study. And maps with the structure of the lifts F may be easier still. If we can use these lifts to say fairly general things about how an f may behave, this would be quite important. For example, this quantity $\deg(f)$ is defined solely by a choice of lift f. We will see just what information $\deg(f)$ conveys. For a moment, let's first take a look at why some of the assertions we just made are true.

- *Lifts always exist.* Simply construct one using the definition. I leave this as an exercise.
- *F is unique up to a constant.*

  Proof. Suppose $\overline{F}$ is another lift. Then

$$\left[\overline{F}(x)\right] = f([x]) = [F(x)], \quad \forall x \in \mathbb{R}.$$

  This is just another way of saying that $\pi \circ \overline{F} = f \circ \pi = \pi \circ F$, $\forall x \in \mathbb{R}$. But then $\overline{F} - F$ ia <u>always</u> an integer! (why?) But $\overline{F} - F$ is the difference between two continuous functions, and hence is continuous. But a continuous function on $\mathbb{R}$ that take values in the integers is necessarily constant. □

- $\deg(f)$ *is well defined.*

  Proof. Here $\deg(f) = F(x+1) - F(x)$ is a continuous function on $\mathbb{R}$ that takes values in the integers (it must, due to the projection π). Thus it also is a constant for all $x \in \mathbb{R}$. □

- *If f is a homeomorphism, then* $|\deg(f)| = 1$.

Proof. Suppose that $|\deg(f)| > 1$. Then $|F(x+1) - F(x)| > 1$. And since $F(x+1) - F(x)$ is continuous, by the Intermediate Value Theorem, $\exists y \in (x, x+1)$ where $|F(y) - F(x)| = 1$. But then $f([y]) = f([x])$ for some $y \neq x$. Thus f cannot be one-to-one and hence cannot be a homeomorphism.

Now suppose that $|\deg(f)| = 0$. Then $F(x+1) = F(x)$, $\forall x$, and hence F is not one-to-one on the interval $(x, x+1)$. But then neither is f, and again, f cannot be a homeomorphism. □

- $F(x) - x\deg(f)$ *is periodic.*

  Proof. It is certainly continuous (why?) To see that it is periodic (of period-1), simply evaluate this function at $x+1$:

$$\begin{aligned} F(x+1) - (x+1)\deg(f) &= (F(x) + \deg(f)) - (x+1)\deg(f) \\ &= F(x) - x\deg(f). \end{aligned}$$

□

Example 2. Let $f(x) = x$. This is the "identity" map on S^1, since all points are taken to themselves. A suitable lift for f is the map $F(x) = x$ on $\mathbb{R}$. To see this, make sure the definition works. Question: Are there any other lifts for f? What about the map $\overline{F}(x) = x + a$ for a a constant? Are there any restrictions on the constant a? The answer is yes. For a to be an acceptable constant, we would need the definition of a lift of be satisfied. Thus

$$\left[\overline{F}(x)\right] = [x + a] = f([x]) = [x].$$

So the condition that a must satisfy is $[x + a] = [x]$ on S^1. Hence, $a \in \mathbb{Z}$. A new question: For a real number $a \notin \mathbb{Z}$, can $\overline{F}(x) = x + a$ serve as a lift of a circle map? What sort of circle map?

Example 3. Let $f(x) = x^n$. Thinking of x as the complex number $x = e^{2\pi i\theta}$, for $\theta \in \mathbb{R}$, then

$$f(x) = f\left(e^{2\pi i\theta}\right) = \left(e^{2\pi i\theta}\right)^n = e^{2\pi i(n\theta)}.$$

Hence a suitable lift is obviously $F(x) = nx$ (I say obviously, since the variable in the exponent <u>is</u> the lifted variable!) Question: This is a degree n map. For which values of n does the map f have an inverse" And what does the map f actually do for different values of n?

Example 4. Let f be a general degree-r map. Then $F(1) - F(0) = r = deg(f)$. Suppose that $F(0) = 0$. Then $F(1) = r$ and if, for example, $r > 1$, it is now easy to see that there will certainly be a $y \in (0,1)$, where $F(y) = 1$. This was a fact that we used in the proof above to show that f cannot be a homeomorphism. In this case, where $r > 1$, at every point in $y \in (0,1)$ where $F(y) \in \mathbb{Z}$, we will have $\pi \circ F(y) = [F(y)] = 0$ on S^1. This won't happen when $r = 1$. When $r = 0$, the map F will be periodic, which is definitely not one-to-one. Question: What happens when $r < 0$? Draw some representative examples to see.

Definition 5. Suppose that $f : S^1 \to S^1$ is invertible. Then

(1) if $\deg(f) = 1$, f is orientation preserving.
(2) if $\deg(f) = -1$, f is orientation reversing.

Recall from Calculus III that orientation is a choice of direction in the parameterization of a space (really, it exists outside of any choice of coordinates on a space, but once you put coordinates on a space, you have essentially chosen an orientation for that space. This is true at least for those spaces that actually are orientable, that is (Moebius Band?) On $\mathbb{R}$, it is the choice of direction for the symbol ">". On a surface, it is a choice of side. In $\mathbb{R}^3$, one can use the Right Hand Rule. Etc. On S^1, orientation preserving really means that after one applies the map, points to the right of a designated point remain on that side. Orientation reversing will flip a very small neighborhood of a point.

Circle maps may or may not have periodic points. And given an arbitrary homeomorphism, without regard to any other specific properties of the map, one would expect that we can construct maps with lots of periodic points of any period. However, it turns out that circle homeomorphisms are quite restricted. because they must be one-to-one and onto, only certain things can happen. To explain, we will need another property of circle homeomorphisms to help us.

Proposition 6. *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism, with $F : \mathbb{R} \to \mathbb{R}$ a lift. Then the quantity*

$$\rho(F) := \lim_{|n| \to \infty} \frac{F^n(x) - x}{n}$$

(1) exists $\forall x \in \mathbb{R}$,
(2) is independent of the choice of x and is defined up to an additive integer, and
(3) is rational iff f has a periodic point.

Given these qualities, the additional quantity $\rho(f) = [\rho(F)]$ is well-defined and is called the *rotation number* of f.

Some notes:

- $\rho(R_\alpha) = \alpha$. (You should be able to actually calculate this directly using the definition.)
- ρ represents in a way the average rotation of points in a circle homeomorphism.
- **Proposition 7.** *If $\rho(f) = 0$, then f has a fixed point.*

  Another way of saying that if there is no average rotation of the circle map, then somewhere a point doesn't move under f. This is like the Intermediate value Theorem on a closed, bounded interval of $\mathbb{R}$ where a map is positive at one end point and negative at the other.

- If f has a q-periodic point, then for a lift F, we have $F^q(x) = x+p$ for some $p \in \mathbb{Z}$. For example, let $f = R_{\frac{6}{7}}$. Then a suitable lift for f would necessarily satisfy $F^7(x) = x+6$, $\forall$x$\in \mathbb{R}$.
- **Proposition 8.** *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism. Then all periodic points must have the same period.*

This last point is quite restrictive. Essentially, if an orientation preserving homeomorphism has a fixed point, it cannot have periodic points of any other period, say. Note that this is not true of a orientation reversing map. For example, the map which flips the unit circle in $\mathbb{R}^2$ across the y-axis, will fix the two points $(0,1)$ and $(0,-1)$, while every other point is of order two.

Exercise 1. For $R_\alpha : S^1 \to S^1$ a circle rotation, show $\rho(R_\alpha) = \alpha$.

Exercise 2. Show that any lift of the rotation $R_{\frac{6}{7}}$ must satisfy $F^7(x) = x + 6$, $\forall$x$\in \mathbb{R}$, and explicitly construct two such lifts.

This is enough for circle homeomorphisms for now. And ends our work in Chapter 4. There is a great section on frequency locking on page 141. Look it over at your leisure. We won't work through it in the course, but it is very interesting. Dynamically, it represents a situation where a linear flow on the torus (with its uncoupled ODEs) becomes the limiting system to a system of coupled ODEs, representing a nonlinear flow. Question: For this to be the case, must the resulting linear flow on the torus be a rational flow?

2. Toral translations

In this short chapter, the only thing I want to discuss is a way to understand toral flows and maps in higher dimensions. For this, let's describe the space. By definition, the n-dimensional torus, or the n-torus, denoted $\mathbb{T}^n$ is simply the n-fold product of n circles

$$\mathbb{T}^n = \overbrace{S^1 \times \cdots \times S^1}^{n \text{ times}}.$$

Think of a system of equations where the n variables are all angular coordinates. Then

$$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = \mathbb{R}/\mathbb{Z} \times \cdots \times \mathbb{R}/\mathbb{Z}.$$

Recall the Kepler Problem. With n point masses, the resulting flow may be seen as linear motion on $\mathbb{T}^n$.

Another way to view the n-torus is via an identification within $\mathbb{R}^n$. Remember the unit square with it opposite sides identified plays a good model for the 2-torus, $\mathbb{T} = \mathbb{T}^2$. The generalization works well here for all the natural numbers. Take the unit cube in $\mathbb{R}^3$. Identify each of the opposite pairs of sides, squares in this case (think of a die, and identify two sides if their numbers add up to 7). The resulting model is precisely the $\mathbb{T}^3$. This works well if one wants to watch a linear flow on $\mathbb{T}^3$. Simply allow the flow to progress in the unit cube, and whenever one hits a wall, simply vanish and reappear on the opposite wall, entering back into the cube.

Note this also works well for $n = 1$: Take the unit interval and identify its two sides (the numbers $x = 0$ and $x = 1$). This is what I mean by the phrase $0 = 1$ on S^1, where the circle is the 1-torus.

Now, the vector exponential map

$$(\theta_1, \dots, \theta_n) \overset{\exp}{\longmapsto} \left(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}\right)$$

maps $\mathbb{R}^n$ onto $\mathbb{T}^n$. We can define a (vector) rotation on $\mathbb{T}^n$ by the vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, where

$$R_{\boldsymbol{\alpha}}(\mathbf{x}) = (x_1 + \alpha_1, \dots, x_n + \alpha_n) = \mathbf{x} + \boldsymbol{\alpha}.$$

Normally, this is called a *translation* (by $\boldsymbol{\alpha}$) on the torus. Note that it should be obvious that if all of the α_i's are rational, then the resulting map on $\mathbb{T}^n$ will have closed orbits. Questions to ask are: Are theses the only periodic linear maps? If one or more α_i's are not rational, can there still be periodic orbits? And if there cannot, are the non-periodic orbits dense in the torus?

Next time, we will answer these questions.