MATH 421 DYNAMICS

Week 6 Lecture 2 Notes

Proposition 1. *if $\gamma = \frac{\omega_2}{\omega_1}$ is irrational, then the flow is minimal. If $\gamma \in \mathbb{Q}$, then every orbit is closed.*

Proof idea. Choose any circle (an easy-to-see choice would be the $x_1 = 0$ circle, which is represented in the plane by the left edge of the unit square). We could call this the waist circle. Then for any point $(0, y)$ on the waist circle, the first-return map of $\mathcal{O}^+_{(0,y)}$ is exactly the rotation map of this circle given by R_γ. As the entire waist circle flows around $\mathbb{T}$, and with the irrational rotation, the orbit $\mathcal{O}^+_{(0,y)}$ will intersect the waist circle densely, the orbit of $(0, y)$ (and hence every point) will be dense in all of $\mathbb{T}$.

For the other statement, simply show that the orbit of every point will eventually return to its starting point, and since the flow is always along straight lines, this is enough to show the periodicity of any and hence all points. □

Example 2. We can look at this another way: Think of $S^1 \in \mathbb{R}^2$ as a circle of radius r centered at the origin. Then we can represent $\mathbb{T}$ as the set

$$\mathbb{T} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \middle| x_1^2 + x_2^2 = r_1^2, \;\; x_3^2 + x_4^2 = r_2^2 \right\}.$$

Now recall a continuous rotation in $\mathbb{R}^2$ is given by the linear ODE system $\dot{\mathbf{x}} = B_\alpha \mathbf{x}$, where B is the matrix $\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$ whose eigenvalues $\pm\alpha i$ are purely imaginary. Do this for each pair of coordinates (each of two copies of $\mathbb{R}^2$) to get the partially uncoupled system of ODEs on $\mathbb{R}^4$,

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \dot{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & -\alpha_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

We will eventually see that this is the model for the spherical pendulum.

Some notes:

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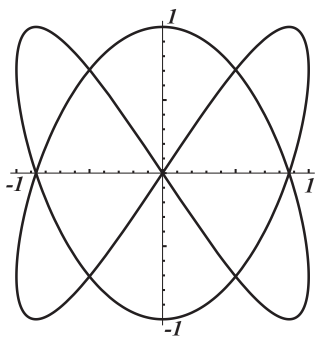
- The two circles $x_1^2 + x_2^2 = r_1^2$ and $x_3^2 + x_4^2 = r_2^2$ are invariant under this flow.
- We can define angular coordinates on $\mathbb{T}$ via the equations
$$x_1 = r_1 \cos 2\pi\varphi_1 \quad x_2 = r_1 \sin 2\pi\varphi_1$$
$$x_3 = r_2 \cos 2\pi\varphi_2 \quad x_4 = r_2 \sin 2\pi\varphi_2.$$
Then, restricted to these angular coordinates and with $\omega_i = \frac{\alpha_i}{2\pi}$, $i = 1, 2$, we recover
$$\dot{\varphi}_1 = -\omega_1, \quad \dot{\varphi}_2 = -\omega_2.$$
Motion is independent along each circle, and the solutions are $\varphi_i(t) = \omega_i(t - t_0)$.
- If $\frac{\alpha_2}{\alpha_1} = \frac{\omega_2}{\omega_1} \notin \mathbb{Q}$, then the flow is minimal.

Exercise 1. Do the change of coordinates explicitly to show that these two interpretations of linear toral flows are the same.

Now, for a choice of $\boldsymbol{\omega}$ and $r_1 = r_2 = 1$, project a solution onto either the (x_1, x_3) or the (x_2, x_4)-planes. The resulting figure is a plot of a parameterized curve whose two coordinate functions are cosine (resp. sine) functions of periods which are rationally dependent iff $\boldsymbol{\omega}$ is rational. In this case, the figure is closed, and is called a Lissajous figure. See the figure below for the case of two sine functions (projection onto the (x_2, x_4)-plane, in this case), where $\omega_1 = 2$ and $\omega_2 = 3$.



$\omega_1 = 2, \quad \omega_2 = 3$

Q. What would the figure look like if ω_1 and ω_2 were not rational multiples of each other?

A nice physical interpretation of this curve is as the trajectory of a pair of uncoupled harmonic oscillators, given by

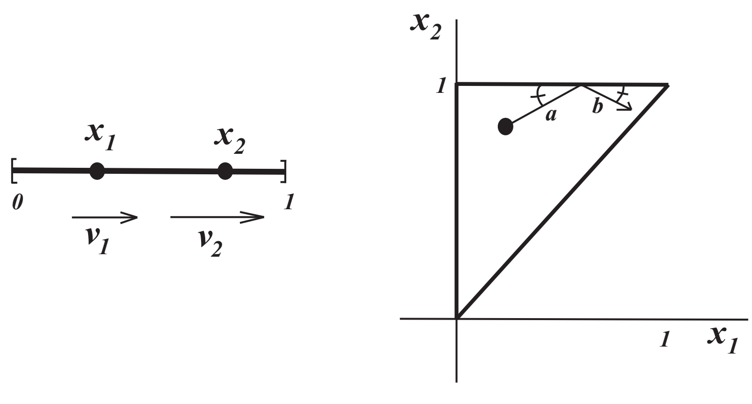
$$\begin{aligned}\ddot{x}_1 &= -\omega_1 x_1\\ \ddot{x}_2 &= -\omega_2 x_2.\end{aligned}$$

Next class, we will begin a study of a related type of dynamical system called a billiard. In one of its most elementary forms, the model of straight-line motion in the plane and its corresponding linear flow on a 2-torus again appear.

Linear flows on the 2-torus: an application (cont'd.)

Besides Lissajous figures, another application of Linear flows on the 2-torus $\mathbb{T}$ involves an area of dynamics called Billiards. Our first example of such a dynamical system uses toral flows directly to give a very strong conclusion.

Consider the unit interval $I = [0, 1]$ with two point masses x_1 and x_2, with respective masses m_1 and m_2 respectively, free to move along I but confined to stay on I. Without outside influence, these point masses will move at a constant, initial velocity. Eventually, they will collide with each other and with the walls. Assume also that theses collisions are elastic, with no energy absorption or loss due to friction. Here, elastic means that, upon a wall collision, a point mass' velocity will only switch sign. And upon a point mass collision, the two point masses will exchange velocities. For now, assume that $m_1 = m_2 = 1$.



The state space in $\mathbb{R}^2$ is

$$T = \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| 0 \leq x_1 \leq x_2 \leq 1 \right\}.$$

Here, T is the region in the unit square above the diagonal line which is the graph of the identity map on I (can you see this?). The edges of the region T are included; since the point masses have no size, they can occupy the same position at the point of contact. An interesting question to ask yourself is: How does the state space change if the point masses had size to them?

Now, given an initial set of data, with initial positions and velocities v_1 and v_2, respectively, what is the evolution of the system? The answer lies in the study of these types of dynamical systems called billiards. Evolution will look like movement in T. A point in T comprises the simultaneous positions of the two particles, and movement in T will consist of a curve parameterized by time t. The idea is that this curve will be a line since the two velocities are constants. The slope of this line (in the figure, line a is the trajectory before any collisions have happened), will be $\frac{v_2}{v_1}$. (why?) Once a collisions happens, though, this changes. There are two types of collisions: Assuming that $\frac{v_2}{v_1}$ is the ratio of the velocities of the two point masses before a collision, we have

- When a point mass hits a wall, it "bounces off", traveling back into i with equal velocity and of opposite sign. Thus the new velocity is $-\frac{v_2}{v_1}$ (This is the slope of line b in the figure above).
- When the two point masses collide, they exchange their velocities (really, think of billiard balls here). Thus the new velocity is $\frac{v_1}{v_2}$. Caution: This reciprocal velocity is NOT the slope of a perpendicular line, which would be the negative reciprocal.

Envision these collisions in the diagram and the resulting trajectory curves before and after each type of collision, as in the figure. What you see are perfect rebounds off of each of the three walls, where the angle of reflection equals the angle of incidence. An ideal billiard table, although one with no pockets. Which leads to the obvious question: What happens if a trajectory heads straight into a corner? For now, we will accept the stipulation that

- When the two point masses collide with a wall simultaneously, either at separate ends of I or at the same end, both velocities switch sign. While this will not change the slope of the trajectory, it will change the direction of travel along that piece of trajectory line.

Some questions to ask:

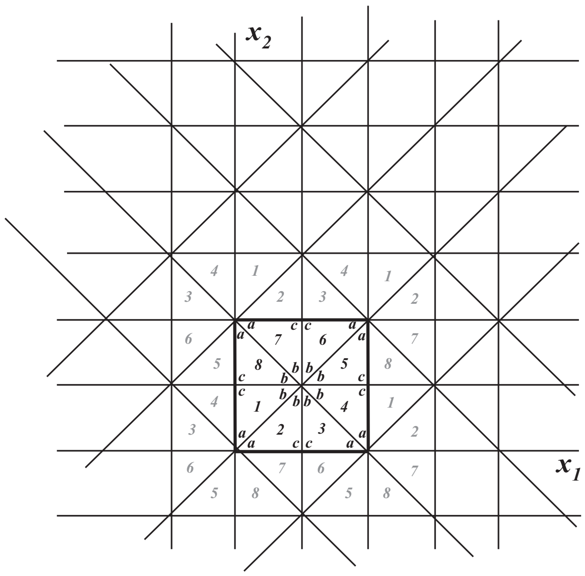
Q. Can there exist closed trajectories?
Q. Can there exist a dense orbit?
Q. The orbits of points in T will very much intersect each other and many trajectories will intersect themselves also. The phase space will get quite messy. Is there a way to better "see" the orbits of points more clearly?

The answer to the last question is yes, although this table is fairly special. Here, one can "unfold" the table:

- Think of the walls of T as mirrors. When a trajectory hits a wall, it rebounds off in a different direction. However, its reflection in the mirror simply continues its straight line motion. Think of a reflected region T across this wall. The trajectory looks to simply pass through the wall and continue on.
- Envision each collision that follows also via its reflection. Motion continues in a straight line fashion through each mirrored wall. By continuing this procedure, the motion will look linear for all forward time, no?
- This idea works because this particular triangle, under reflections, will eventually cover the plane in a way that only its edges overlap and all points in $\mathbb{R}^2$ are covered by at least one triangle. This is called a *tiling* of the place by T, and works only because T has some special properties. See below.
- The unfolded trajectory is called a linear flow on the billiard table $\mathbb{R}^2$.

So what does a billiard flow in $\mathbb{R}^2$look like? Obviously, it is just straight line motion at a slope $\frac{v_2}{v_1}$ forever since there are no collisions. The better question to ask is: What does this tell us about the original flow on the triangle T?

By continually unfolding (reflecting) the table T, on starts to notice that there are only 8 different configurations: the four orientations of T given by rotations by multiples of $\frac{\pi}{2}$ radians, and the reflection of each. If you collect up a representative of each of these configurations into a connected region, you wind up with enough information to characterize the entire flow in $\mathbb{R}^2$: Each time your $\mathbb{R}^2$ linear flow re-enters a region of a particular configuration of T, you can simply note the trajectory in your representative of that region. This region of representative configurations is called a *fundamental domain* for the flow. One such fundamental domain or this flow is the square of side length 2 in the figure. Noting the configurations, as the trajectory leaves the square, it enters a configuration exactly like that at the other side of the square. One can see the trajectory then re-enter the square from the other side. Similarly, when one leaves the square at the top, it enters a configuration represented at the bottom of the square. Thus one can continue the trajectory as if it had re-entered the square at the bottom.



Note: There was a famous arcade video game from the Middle Ages (you know, like, the 80's!!) where a space ship was planted in the middle of a square screen. It could turn but not move. Various boulders (asteroids, actually: this was the name of the game) would float in and out of the screen. Should an asteroid hit the ship, the game is over. The ship can fire a weapon at an asteroid, and if hit, would break into two smaller ones, which would go off in different directions. The asteroids (or pieces of asteroids) always traveled in a straight line. And as an asteroid left the screen, it would always reappear on the opposite side and travel in the same direction. Really, the asteroids were only exhibiting a linear toral flow. Who would have though that in playing this game, one was actually playing in a universe which was not the plane at all but rather the torus $\mathbb{T}$?

Hence linear flows on $\mathbb{R}^2$ again look like toral flows on this fundamental domain, which comprises the space of configurations of T as one uses T to tile $\mathbb{R}^2$. SO what do linear toral flows say about the trajectories on T?

Proposition 3. *If the ratio of initial velocities $\frac{v_2}{v_1} \in \mathbb{Q}$, then the orbit is closed (on $\mathbb{T}$ and thus also on T). If $\frac{v_2}{v_1} \notin \mathbb{Q}$, then the orbit is dense in T.*

Note: Now, it is easier to see what a collision in a corner will look like. Like I said, this table T is quite special in many ways. These ways do not generalize well. However, with T we can say much more:

Proposition 4. *For any starting set of data (point mass positions and velocities), the trajectory will assume at most* 8 *different velocity ratios.*

Count them: There are two possible ratio magnitudes, each with two signs. That makes 4. But travel along the lines of each of these slopes can be in each of the two directions due to the reflected configurations in the fundamental domain.

How can one generalize these results to other tables:

- **Unequal masses.**
  - An elastic collision between unequal masses will not result in what would look like a reflection off of the diagonal wall in T. One could certainly accurately chart the collision as a change in direction off of the wall. However, when unfolding the table, the resulting flow in $\mathbb{R}^2$ will not be linear (each reflected trajectory through the diagonal wall will be a change in direction in the planar flow. You will see a piecewise linear flow in $\mathbb{R}^2$ and hence also on the fundamental domain $\mathbb{T}$. While this is workable, it is not such as easy leap to a conclusion.
  - One can also actually change the table. Use momenta to define the collision between the point masses, and alter the diagonal wall to be a perfect reflective wall. The resulting will not be linear. The

new table will not tile the plane anymore, but in many cases the unfolded table will cover the plane with many holes (the reflecting curve will be concave, so will fit into the original T. The unfolded flow will look liner until it hits a hole, where it will reflect through he hole perpendicularly through its center axis and appear on the other side to continue at the same slope. I haven't worked out the details here (and a hat tip to Jonathan Ling who started to work on this idea), but there should be results here that are similar to the original table T, as long as one is careful with the analysis.

- **Other tilings of $\mathbb{R}^2$.** It is easy to see that some shapes tile the plane while others do not. Rectangles, and a few other triangles work fine. And a few other polygonal shapes, like regular hexagons also. There are some examples in the book. But examples are fairly rare. And in each case, one would need to find a fundamental domain and then interpret the resulting flow on that domain in terms of the original flow as well as that on the place. All good stuff, and are the initial ways one may study polygonal billiards. However, later, we will generalize our analysis of billiards in a completely different direction.