MATH 421 DYNAMICS

Week 6 Lecture 1 Notes

Last week's discussion on irrational rotations of S^1 introduced a new type of orbit behavior, where an orbit can densely fill a space. To continue this discussion, we will need to name this type of behavior and try to characterize why some dynamical systems behave this way. We start today with a few definitions.

Definition 1. A set $Y \subset X$ is invariant under a map $f : X \to X$, if

$$f\Big|_Y : Y \to Y.$$

Definition 2. A homeomorphism $f : X \to X$ is called *topologically transitive* if $\exists x \in X$ such that $\mathcal{O}_x$ is dense in X. An non-invertible map is called topologically transitive if $\exists x \in X$ such that $\mathcal{O}_x^+$ is dense in X.

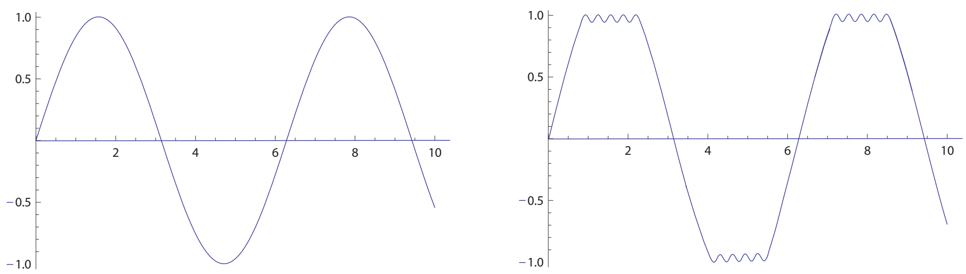
Definition 3. A homeomorphism $f : X \to X$ is *minimal* if $\forall x \in X$ $\mathcal{O}_x$ is dense in X (the forward orbit is dense for a noninvertible map).

Definition 4. A closed, invariant set is *minimal* is there does not exist a proper, closed invariant subset.

More notes:

- Like in the case of open and closed domains in vector calculus, a set is closed if it contains all of its limit points. And for any set X, the closure of X, denoted $\overline{X}$ is defined to be the closed set obtained by adding to X all of its limit points (think of adding the sphere which is the boundary of an open ball in $\mathbb{R}^3$). In the case of a minimal map $f : X \to X$, for any $x \in X$, we have $\overline{\mathcal{O}_x} = X$.
- Same is true for a topologically transitive map f, if one takes any point on the dense orbit.
- Irrational rotations of the circle are minimal!.

Date: March 8, 2013.



0.1. **Application: Periodic Function Reconstruction via Sampling.** Consider the two functions in the picture.

- Each is periodic and of the same period as the other.
- Each can be viewed as a real-valued smooth function on S^1. And each takes values in the interval $I = [-1, 1]$.
- Question: Are the values of these two functions equally distributed equally (or even evenly) on I?
- Question: If we knew the period and range of some unknown function, and needed to sample the function (create a sequence of function values) to see which of the above two function was the one we are seeking, how can we design our sampling to ensure we can differentiate between these two?

Dynamics attempts to answer this question. Let $\{x_n\}$ be a sequence (think of this sequence as a sampling of the function), and $a < b$ two real numbers. Define

$$F_{a,b}(n) = \#\left\{k \in \mathbb{Z} \middle| 1 \leq k \leq n, a < x_k \leq b\right\}$$

as the number of times the sequence up to element n visits the interval $(a, b) \subset \mathbb{R}$. Really, this is the same definition of F as before on the arc $\Delta \subset S^1$. The only change in this case is that we are defining F in this context as an interval in $\mathbb{R}$. Then define the *relative frequency* in the same way as before. In the figure, the relative frequency of $\{x_n\}$ on the interval $(a, b]$

shown is

$$\left.\frac{F_{a,b}(n)}{n}\right|_{n=6} = \frac{2}{6} = \frac{1}{3}.$$

We say that $\{x_n\}$ has an *asymptotic distribution* if $\forall a, b$, where $-\infty \leq a < b \leq \infty$, the quantity

$$\lim_{n\to\infty} \frac{F_{a,b}(n)}{n}$$

exists. In a sense, we are defining the percentage of the time that a sequence visits a particular interval.

In the case where the sequence has an asymptotic distribution, the function

$$\Phi_{\{x_n\}}(t) = \lim_{n\to\infty} \frac{F_{-\infty,t}(n)}{n}$$

is called the distribution function of the sequence $\{x_n\}$. Here Φ is monotonic, and measures how often the values of a sequence visit regions of the real line as one varies the height of an interval $(-\infty, t]$.

Definition 5. A real-valued function φ on a closed, bounded interval is called *piecewise monotonic* if the domain can be partitioned into finite many subintervals on which φ is monotonic. A real-valued function on $\mathbb{R}$ is *piecewise monotonic* if it is piecewise monotonic on every closed, bounded subinterval of $\mathbb{R}$.

Remark 6. Monotonic means strictly monotonic here. Really, this means that there are no flat (purely horizontal on an open interval) regions of the graph of φ. Think of functions like $f(x) = \sin x$, and polynomials of degree larger than 1, which are piecewise monotonic, and functions like

$$g(x) = \begin{cases} -(x+2)^2 & -4 \leq x < -2 \\ 0 & -2 \leq x \leq 0 \\ x^2 & 0 < x \leq 2 \end{cases},$$

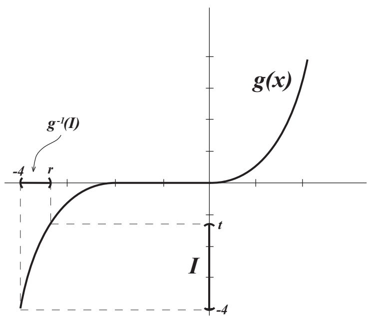
which is not piecewise monotonic (See the graph of $g(x)$ below).

When φ is piecewise monotonic, the pre-image of any interval I is a finite union of intervals in the domain (see the figure).

Definition 7. The φ-length of an interval I is

$$\ell_\varphi(I) := \ell\left(\varphi^{-1}(I)\right).$$

- This is the total length of all pieces of the domain that map onto I. In the figure, $\ell_\varphi(I) = \ell(A) + \ell(B)$.
- For piecewise monotonic functions φ, the φ-length is a continuous function of the end points of I (vary one end point of I continuously, and the φ-length of I also varies continuously. This doesn't work with flat regions since the φ-length ell_φ would then jump as one hits the value of the flat region.



Indeed, let's look at the $g(x)$ in the figure more closely. Here, one can calculate the φ-length. Indeed, choose the interval $I = [-4, t]$. Here, t is the function value, and there is only a single interval mapped onto i for any value of t.

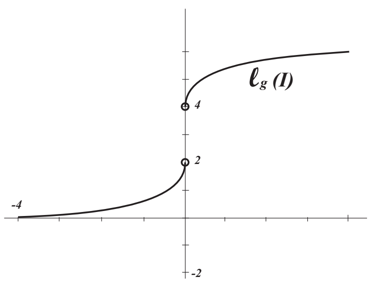
For $t < 0$, this interval is given in the figure as the interval of the domain $g^{-1}(I) = [-4, r]$, where $g(r) = t$. Solving the equation $g(r) = t$ for r yields

$$-(r+2)^2 = t \iff r = -\sqrt{-t} - 2$$

where we chose the negative branch of the square root function in the middle step to account for the domain restrictions. Here, the g-length of I,

$$\begin{aligned} \ell_g(I) &= \ell\left(g^{-1}([-4,t])\right) \\ &= -2 - \sqrt{-t} - (-4) = 2 - \sqrt{-t}. \end{aligned}$$

Now for $t > 0$, the same calculation yields $\ell_g(I) = 4 + \sqrt{t}$ for $I = [-4, t]$. Putting these two pieces of the g-length function together yields the graph of



$$\ell_g(I) = \begin{cases} 2 - \sqrt{-t} & -4 \le t < 0 \\ 4 + \sqrt{t} & 0 < t \le 4 \end{cases}$$

which has a jump discontinuity at $t = 0$. In fact, the only way to change $g(x)$ to make the g-length function continuous is to remove the middle piece of the $g(x)$ function and translate one or the other pieces right or left to again make $g(x)$ continuous. But that would have the effect of moving the two pieces of the graph of $\ell_g(I)$ together. The jump discontinuity becomes a hole in the graph, easily filled. But in this case, the changed $g(x)$ has been made piecewise monotone!

One can show that for a piecewise monotonic function φ, a distribution function for φ is

$$\Psi : \mathbb{R} \to \mathbb{R}, \quad \Psi_\varphi(t) = \ell_\varphi\left((-\infty, t)\right).$$

We can use this for:

Theorem 8. *Let φ be a T-periodic function of $\mathbb{R}$ such that $\varphi_T = \varphi|_{[0,T]}$ is piecewise monotone. If $\alpha \notin \mathbb{Q}$ and $t_0 \in \mathbb{R}$, then the sequence $x_n = \varphi(t_0 + n\alpha T)$ has an asymptotic distribution with distribution function*

$$\Phi_{\{x_n\}}(t) = \frac{1}{T}\Psi_\varphi(t) = \frac{\ell\left(\varphi^{-1}\left((-\infty, t)\right)\right)}{T}.$$

We won't prove this or study it in any more detail. But there is an interesting conclusion to draw from this. In the theorem, the sequence of samples of the T-periodic function φ has the same distribution function as the actual function φ, (defined over the period, that is) precisely when the sampling is taken at a rate which is an irrational multiple of the period T. In this way, the sequence, over the long term, will fill out the values of φ over the period in a dense way. In a way, one can recover the function φ from a sequence of regular samples of it only if the sampling is done in a way which ultimately allows for all regions of the period to be visited evenly. This is a very interesting result.

In the book is an actual calculation of the distribution function for the sequence $\{\sin n\}$. Since the natural numbers are not a rational multiple of 2π, the period of the sine function, this distribution function is precisely the same as that distribution function of the smooth function $f(x) = \sin x$, defined on the interval $[0, 2\pi]$. Take a good look at this example.

0.2. **Application: Linear Flows on the 2-Torus.** Here is another application of circle rotations and their implications. This one involves generalizing circle maps via a corresponding circle flow into more than one dimension.

For this application, we will skip Section 4.2.2 on the distribution of first digits of powers, and proceed to Section 4.2.3. To start, however, recall what a flow is: Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$ be an IVP, where the vector field $\mathbf{f}(\mathbf{x}) \in C^1$. This IVP defines a flow on $\mathbb{R}^n$. For $I \subset \mathbb{R}$ an interval containing 0, define a continuous map $\varphi : I \times \mathbb{R}^n \to \mathbb{R}^n$ that satisfies the following:

- $\forall T \in I$, $\varphi^t = \varphi(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism (for a given choice of t, this is is simply the time-t map of the IVP).
- $\forall s, t \in I$, where $s + t \in I$, one has

$$\varphi^s \circ \varphi^t(\mathbf{x}) = \varphi^{s+t}(\mathbf{x}).$$

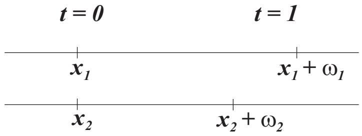
Now suppose that $S^1 = \{e^{2\pi i x} \in \mathbb{C}\}$, and $\frac{dx}{dt} = \alpha$, $x(0) = x_0$ is an IVP defined on S^1. This is solved by $x(t) = \alpha t + x_0$, which can also be written in flow form $\varphi^t_\alpha(x) = \alpha t + x$. Notice in this last expression, we have included the subscript α to denote the dependence of the flow on the value of the parameter α. here the time-1 map is just

$$\varphi^1_\alpha(x) = \alpha + x = R_\alpha(x), \quad x \in S^1.$$

The time-1 map is just a rotation map of the circle by α. Keep in mind, however, that the IVP will share the same time 1 map as the new IVP given by $\frac{dx}{dt} = \alpha + 1$, $x(0) = x_0$. However, the original flows are very different! Linear flows on S^1 are not very interesting. They differ only by speed (and possibly direction), and ultimately, all look like continuous rotations of the circle, whether α is rational or not. However, we can generalize this flow to a situation which does produce somewhat interesting dynamics.

Consider now a flow given by the pair of uncoupled circle ODEs:

$$\frac{dx_1}{dt} = \omega_1, \quad \frac{dx_2}{dt} = \omega_2.$$



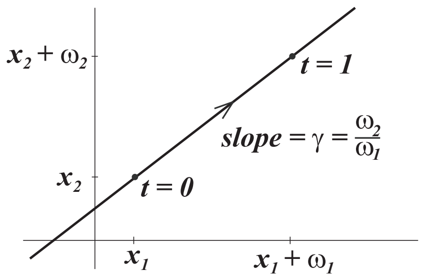
This system, which can be written as the uncoupled vector ODE $\dot{\mathbf{x}} = \boldsymbol{\omega}$, or $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$, can be viewed as defining a flow on the two-torus $\mathbb{T} = S^1 \times S^1$, and has the solution

$$\mathbf{x} = \begin{bmatrix} x_1 + \omega_1 t \\ x_2 + \omega_2 t \end{bmatrix}.$$

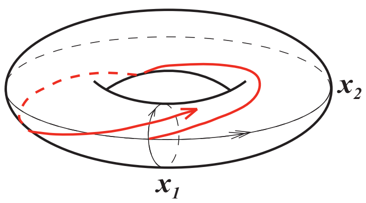
In flow notation, we can write either

$$T^t_{\boldsymbol{\omega}}(x_1, x_2) = \big(x_1 + \omega_1 t, x_2 + \omega_2 t\big)\,, \text{ or } \varphi^t_{\boldsymbol{\omega}}(\mathbf{x}) = \mathbf{x} + \boldsymbol{\omega} t.$$

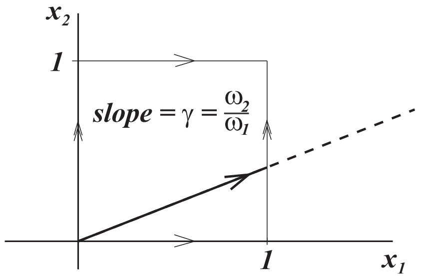
Graphically, solutions are simply translations along $\mathbb{R}$ or as straight line motion in $\mathbb{R}^2$. Note that in this last interpretation, the slope of the solution line is $\gamma = \frac{\omega_2}{\omega_1}$.

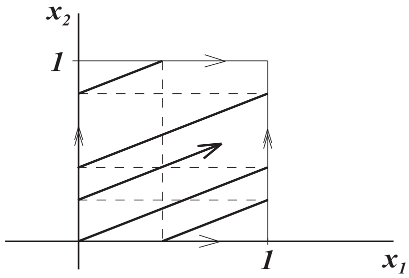


However, each of these uncoupled ODEs also can be considered as a flow on S^1, and hence the system can be considered a flow on $S^1 \times S^1 = \mathbb{T}$. Suppose, for example, that $1 < \omega_1 < 2$, while $0 < \omega_2 < 1$. The flow from time $t = 0$ to time $t = 1$ would take the origin on one circle to the point $1 - \omega_1$, and the flow line would start at $x_1 = 0$ and travel once around the circle before stopping to ω_1. The flow on the other circle would take $x_2 = 0$ partway around the circle to ω_2. Viewed via the two periodic coordinates of $\mathbb{T}$, we have the flow line in the picture:



Another way to see this is to go back to the plane and consider the equivalence relation given by the exponential map on each coordinate. The set of equivalence classes are given by the unit square in the plane, under the idea that the left side of the square (the side on the $x_1 = 0$ line) and the right side (the $x_1 = 1$ side) are considered the same points (this is the $0 = 1$ idea of the circle identification). Similarly, the top and bottom of the square are to be identified. Then the flow line at the origin under the ODE system is a straight line of slope γ emanating from the origin and meeting the right edge of the unit square at the point $(1, \gamma)$. But by the identification, we can restart the graph of the line at the same height on the left side of the square (at the point $(0, \gamma)$. Continuing to do this, we will eventually reach the top of the square. But by the identification again, we will drop to the bottom point and continue the line as before. In essence, we are graphing the flow line as it would appear on the unit square. When we pull this square out of the plane and bend it to create our torus $\mathbb{T}$, the flow line will come with it. Suppose $\gamma \notin \mathbb{Q}$. What can we say about the positive flow line?





Next class, we will continue this discussion with a few conclusions about this flow.