MATH 421 DYNAMICS

Week 5 Lecture 2 Notes

Recurrence and more complicated behavior

So far, we have explored many systems and contexts where dynamical systems have exhibited simple behavior, or fairly simple behavior. We will now begin to explore more complicated behavior than before. However, to start, we will stay with maps of the type you have already played with. But we will change the place on which they are acting. This, in and of itself, changes the nature of the orbits. It turns out that when the space is Euclidean, orbits can converge to something or wander away toward the edge of the space. However when a space is compact, roughly that its edges are not infinitely far away, and the edges are in fact in the space, then an orbit that does not converge to any particular thing must go somewhere within the space. How to describe where it goes will take us to behavior which is more complicated than what we have already seen. To begin, consider the definition:

Definition 1. For $f : X \to X$ a continuous map on the metric space X, a point $x \in X$ is called (*positively*) *recurrent* with respect to f if there exist a sequence of natural numbers $n_k \longrightarrow \infty$ where $f^{n_k}(x) \longrightarrow x$.

In the simple dynamical systems we studies so far, the only recurrent points were fixed and periodic points (this makes sense, right?). However, non-periodic points an also be recurrent. This chapter begins a study of relatively simple maps that exhibit much more complicated behavior. And this behavior is captured in this notion of recurrence.

Rotations of the circle

Again, think of S^1 either as the set of unit modulus numbers of the complex plane

$$S^1 = \left\{ z \in \mathbb{C} \middle| z = e^{2\pi i\theta}, \theta \in \mathbb{R} \right\},$$

or as the quotient space of the real line modulo the integers, $S^1 = \mathbb{R}/\mathbb{Z}$. Recall, for $x, y \in \mathbb{R}$, denote $\overline{x}$, $\overline{y}$ their respective points in S^1 under the exponential map $\rho : \mathbb{R} \to S^1$, $\rho(\theta) = e^{2\pi i\theta}$.

- Here $\overline{x} = \overline{y}$ iff $x - y \in \mathbb{Z}$, or $x \equiv y \pmod 1$.
- $\overline{x}$, $\overline{y}$ are the equivalence classes of points in $\mathbb{R}$ under the equivalence relations imposed on $\mathbb{R}$ by the map ρ.

In this last interpretation, one can imagine S^1 to be the unit interval $[0, 1]$ in $\mathbb{R}$ where one agrees to identify the endpoints (hence the notation I sometimes use in class that $0 = 1$).

One can define a metric on S^1 by simply inheriting the one it has as it sits in $\mathbb{C}$ (or if you will, $\mathbb{R}^2$). This is essentially the Euclidean metric and measures the straight line distance in

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the plane between two points. Really, this is the length of the chord, or secant line, joining the points. See the figure. But also, we can define a distance between points by the arc length between them. In some ways, this is preferable, since in the abstract, S^1 doesn't really sit anywhere. There is no interior and exterior of S^1, unless you call the actual points in the curved line making the circle the interior points. The problem with using arc length to determine the distance between points is that there are two distinct paths going from one point to another. There must be a determination as to which one to choose. Choosing the minimal path is a nice choice, but how does one do this mathematically. The answer lies within the view that S^1 really is the real line $\mathbb{R}$ infinitely coiled up like a slinky by the exponential map ρ above, and length in $\mathbb{R}$ is easy to describe, and passes through this map, at least locally:

Define

$$d\left(\overline{x},\overline{y}\right) = \min\left\{ |x-y| \,\middle|\, x,y \in \mathbb{R}, x \in \overline{x}, y \in \overline{y} \right\}.$$

The figure below shows the equivalence classes of the points $\overline{x} = \frac{1}{3}$ and $\overline{y} = \frac{3}{4}$. Choosing arbitrary representatives x and y and calculating their distance in $\mathbb{R}$ will lead to many different results. However, the minimum distance between representatives of these two classes is well-defined and in this case, $d\left(\overline{x},\overline{y}\right) = \frac{5}{12}$. Notice that really, the closest two distinct distances between two equivalence classes in $\mathbb{R}$ correspond precisely to the arc lengths in S^1 along the two distinct paths joining $\overline{x}$ and $\overline{y}$.

Lemma 2. *These two metrics are equivalent.*

Proof. This is a really good exercise. □

Denote by R_α the rotation of S^1 by the angle α. We have parameterized S^1 as the unit interval in $\mathbb{R}$, with $0 = 1$. So even though α technically can be any real number, rotating by α and rotating by $\alpha + n$, where $n \in \mathbb{Z}$ amounts to the same thing. (Note that this would definitely not be the case for a continuous dynamical system given by $\dot{x} = \alpha x$, $x \in S^1$. Can you see why?) Here $R_\alpha(\overline{x}) = \overline{x} + \alpha$. In complex notation, we view rotations as linear maps, with multiplication by the factor $z_\alpha = e^{2\pi i \alpha}$, so that $R_\alpha(z) = z_\alpha z$. In each case, then

$$R_\alpha : S^1 \to S^1, \text{ with either } R_\alpha^n(\overline{x}) = \overline{x} + n\alpha \text{ or } R_\alpha^n(z) = z_\alpha^n z.$$

Q. What can we say about the dynamics of a circle rotation?
Q. What if $\alpha \in \mathbb{Q}$?
Q. What if $\alpha \notin \mathbb{Q}$?

The quick answers are that, when α is rational, all orbits are periodic, and all of the same period. When α is not rational, then there are no periodic orbits at all. I ask you to show this in the exercises, and the trick really is to understand well what R_α^n looks like for each n, and what it means for a point to be periodic in the circle.

Exercise 1. Let $R_\alpha : S^1 \to S^1$, be the rotation $r_\alpha(\overline{x}) = \overline{x} + \alpha$. Show that every orbit is periodic when $\alpha \in \mathbb{Q}$, and no orbit is periodic when $\alpha \notin \mathbb{Q}$.

The latter exercise creates a deeper concern: Without fixed or periodic points in S^1 for what I will call an *irrational rotation*, the question is, where do the orbits go? They cannot converge to a point in the circle, since in many cases (and really in general), if they converged to a point in S^1, then that point would have to be a fixed point (if orbits converge, they must converge to another orbit). The answer is that they go everywhere. And that tells one a lot about the dynamics.

Remark 3. The above notion of an irrational rotation was based on the parameterization of S^1 given by the interval $[0, 1)$. There, the rotation R_α was irrational as a rotation when α isn't rational as a number. However, the parameterization is critical here, and the rationality IS of the rotation really with respect to the integer 1, the maximum value of the parameter. To see this, suppose instead we parameterized S^1 via the interval $[0, 2\pi)$, another rather common parameterization given by the map $\rho : \mathbb{R} \to S^1$, where $\rho(x) = e^{ix}$. Here, a rotation half way around the circle is given by R_π, where $\alpha = \pi$ is irrational (as a number!) Thus the rotation R_π is not irrational at all, as every point is 2-periodic. However, the rotation by 1, R_1 would have NO periodic orbits (show this!). The correct conclusion to draw here is that the rationality of the rotation R_α depends on the parameterization. We offer a definition to be clear.

Definition 4. A rotation $R_\alpha : S^1 \to S^1$, where S^1 is parameterized by the interval $[0, T)$ for $T > 0$, is called *irrational* if $\frac{\alpha}{T} \notin \mathbb{Q}$. Otherwise, the rotation is called *rational*.

Proposition 5. *For R_α an irrational rotation of S^1, all orbits are dense in S^1.*

(idea of proof). Really, the idea is the following:

- Show the forward orbit of any $\overline{x}$ is not periodic (you will do this in the exercises).
- Show that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, such that $d(R_\alpha^N(\overline{x}, \overline{x}) < \epsilon$.
- Show that this is true for all $\overline{x}$.

□

Note: All rotations are invertible, right? Really, they are all homeomorphisms. So define $R_\alpha^{-1}(\overline{x}) = R_{-\alpha}(\overline{x})$. To show density, we have to show that the orbit of $\overline{x}$ will visit any size open neighborhood of $\overline{x}$. Here is a nice technique for showing this:

Continued Fraction Representation. The continued fraction representation (CFR) of a real number is a representation of real numbers as a sequence of integers in a way which essentially determines the rationality of the number. This is very much like the standard decimal representations of real numbers, in that it also (our usual base-10 version is a good example) provides a ready way to represent all real numbers. However, the sequence of integers which represent a real number in a base-10 decimal expansion represent some rational numbers as finite-length sequences (think $\frac{11}{8} = 1.375$), and others as infinite length sequences (think $\frac{4}{9} = 0.44444\cdots$). The CFR instead is a base-free representation in which all and only rational number representations are the finite length sequences. Plus, the CFR is another nice way to approximate a real number by either truncating its sequence or simply not calculating the entire sequence.

Indeed, in the CFR, Any real number in $(0,1)$ can be written as $\frac{1}{s}$, where $s \in (1,\infty)$. More generally, then, any real number r can be written as an integer and a real number in $(0,1)$; as

$$r = n + \frac{1}{s}, \text{ where } n \in \mathbb{Z}, \text{ and } s \in (1,\infty).$$

If $s \in \mathbb{N}$, then this expression is considered the CFR of r (it is sometimes written then $r = [m; s]$; For example, $\frac{5}{2} = [2:2]$.

Now suppose $s \notin \mathbb{N}$. Then since $s \in (1,\infty)$, $s = m + \frac{1}{t}$, for $m \in \mathbb{N}$, and $t \in (1,\infty)$. Thus,

$$r = n + \frac{1}{\frac{1}{m+\frac{1}{t}}}, \text{ where } n \in \mathbb{Z}, \ m \in \mathbb{N}, \text{ and } t \in (1,\infty).$$

Again, if $t \in \mathbb{N}$, then we stop and $r = [n; m, t]$ is the CFR of r. If it is not, we again let $t = p + \frac{1}{u}$, for $p \in \mathbb{N}$ and $u \in (1,\infty)$ so

$$r = n + \frac{1}{\frac{1}{m+\frac{1}{p+\frac{1}{u}}}}, \text{ where } n \in \mathbb{Z}, \ m,p \in \mathbb{N}, \text{ and } u \in (1,\infty).$$

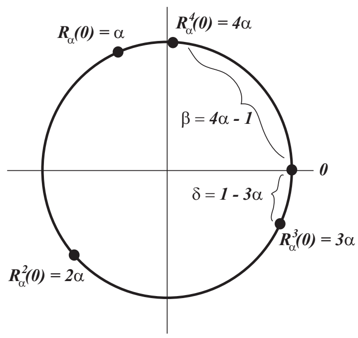
Again, if $u \in \mathbb{N}$, we stop and the CFR of r is $[n : m, p, u]$. If not, then we continue indefinitely. The CFR is a finite sequence iff $r \in \mathbb{Q}$.

Exercise 2. Compute the CFR of $-\frac{33}{13}$.

Exercise 3. Calculate the fraction whose CFR is $[0 : 3, 5, 7]$.

Example 6. So let R_α be a rotation of S^1 for $\alpha = \frac{1}{3+\frac{1}{5+\frac{1}{c}}}$, where $c > 1$, and $c \notin \mathbb{Q}$. Then it turns out that $\alpha \notin \mathbb{Q}$.

To see this, let's start the construction which would establish the middle bullet point in the above proof idea. To start, it should be obvious that $\frac{1}{4} < \alpha < \frac{1}{3}$ (why?). In the figure, we can graph $R_\alpha(0)$, $R_\alpha^2(0)$, $R_\alpha^3(0)$, and $R_\alpha^4(0)$. One of the latter two winds up being the early, closest approach to 0 of the orbit $\mathcal{O}_0^+$. But which is smaller, $\delta = 1 - 3\alpha$, or $\beta = 4\alpha - 1$?



Visually, the closest approach to 0 is $R_\alpha^3(0) = 3\alpha$, but without the benefit of knowing what the choice of c is in general, it is not clear *a priori* whether $\delta = 1 - 3\alpha$ is actually smaller than $\beta = 4\alpha - 1$. Even without knowing c, we can still perform the comparison via the CFR:

$$\delta = 1 - 3\alpha = 1 - \frac{3}{3 + \frac{1}{5+\frac{1}{c}}} = \frac{3 + \frac{1}{5+\frac{1}{c}} - 3}{3 + \frac{1}{5+\frac{1}{c}}} = \frac{\frac{1}{5+\frac{1}{c}}}{3 + \frac{1}{5+\frac{1}{c}}} = \frac{1}{16 + \frac{3}{c}}.$$

Exercise 4. Calculate $\beta = 4\alpha - 1$ in the same way as above, and show that it is larger than δ for any choice of $c > 1$.

Hence, the third iterate is the first closest return of $\mathcal{O}_0^+$ to 0.

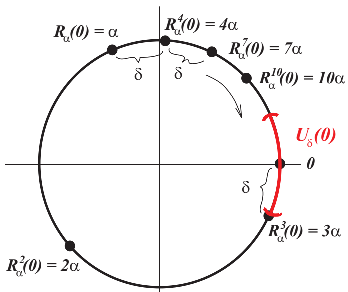
Q. Will the orbit ever get closer to 0?
Q. If it will, then which iterate?

These questions will help us to show the orbit will eventually get arbitrarily close to 0.

We could simply hunt for the next return. Or we can be clever and calculate it. Here is the idea: it took three steps to get within δ of the initial point 0. (We could say it took three steps to get δ-close to 0). If we now create an open δ-neighborhood of 0, $N_\delta(0)$, when will the first iterate occur when we will enter this neighborhood and thus get closer than δ to 0?

One way to ensure this is to look at the first step after our previous close approach. This is the fourth element of $\mathcal{O}_0$ and is $R_\alpha^4(0) = 4\alpha$. Here $4\alpha = \alpha + 3\alpha = \alpha + (1-\delta)$, so that $4\alpha - 1 = \beta = \alpha - \delta$. One conclusion to draw from this is that $R_{3\alpha}$ takes α to 4α which is $\alpha - \delta$ (see figure). So $R_{3\alpha}(\alpha) = \alpha - \delta$, $R_{3\alpha}^2(\alpha) = \alpha - 2\delta$, and $R_{3\alpha}^n(\alpha) = \alpha - n\delta$. So for which n would we satisfy

$$0 < \alpha - n\delta < \delta\ ?$$

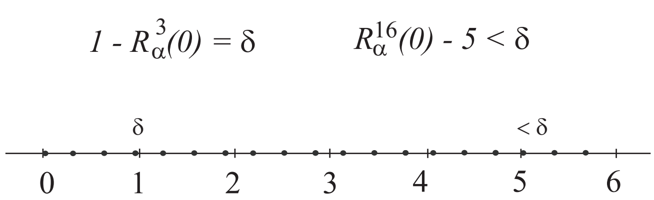


Note that for some choice of n, the iterate will have to lie on the positive side of 0 in $N_\delta(0)$ (why?). Of course, this simplifies to $n\delta < \alpha < (n+1)\delta$, which is solved by simply taking the integer part of the fraction $\frac{\alpha}{\delta}$. Denote the greatest integer function by $\lfloor \cdot \rfloor$, so that, for example, $\lfloor \pi \rfloor = 3$. Then, the iterate n we are looking for is

$$n = \left\lfloor \frac{\alpha}{\delta} \right\rfloor = \left\lfloor \frac{\frac{1}{3+\frac{1}{5+\frac{1}{c}}}}{\frac{1}{16+\frac{3}{c}}} \right\rfloor = \left\lfloor 5 + \frac{1}{c} \right\rfloor = 5.$$

Hence we can say that $R_{3\alpha}^5(\alpha) = R_\alpha^{15}(\alpha) = R_\alpha^{16}(0)$ is within δ of 0 (See figure at right below). We could then use the actual distance between 0 and $R_\alpha^{16}(0)$ as our new δ, and look for iterates of R_α^{16} to find our next closest approach. Continuing this way, we create a subsequence of $\mathcal{O}_0$ which consists of exponentially increasing powers of the original R_α and this subsequence converges to 0. This is the basic approach to proving the second bullet point in the above proof idea.

On the real line, we see that our rotations by α is simply a translation by α. Approaching and getting closer to 0, means that our orbit will at some point come close to an integer value (ANY integer will do, as they all represent 0 in the circle!). See the figure here.



There is really a better way to understand this notion of visiting neighborhoods of points in S^1 under irrational rotations. This other way is by understanding the frequency with which an orbit visits a small open set under a rotation. This is called the *dynamical frequency*, and is a measure of how often an orbit visits a small open interval in S^1 relative to how much time it is outside of the interval.

Fix $\Delta \subset S^1$ an arc. Then for $x \in S^1$ and $n \in \mathbb{N}$, define

$$F_\Delta(x,n) = \#\left\{k \in \mathbb{Z} \middle| \, 0 \le k < n, R_\alpha^k(x) \in \Delta\right\}.$$

Here, the number sign # denotes the cardinality of the set. For example, in the above figure with our choice of α, and $\Delta = N_\delta(0)$, we have

$$F_\Delta(0,18) = F_{N_\delta(0)}(0,18) = \#\left\{0, 16\right\} = 2.$$

Note that for Δ small, then for any $x \in S^1$, F_Δ will be small. And for Δ large, F_Δ will be bigger, but always less than n. So we can say that $0 \le F_\Delta(x,n) \le n$, for every x and Δ. And for any choice of x and Δ, as n grows, F_Δ is monotonically increasing.

However, it is also true that for $\alpha \notin \mathbb{Q}$, $\lim\limits_{n\to\infty} F_\Delta(x,n) = \infty$. (Can you show this?) Hence instead of studying the frequency with which the orbit of a point visits an arc, we study the *relative frequency* of visits as n gets large, or the quantity

$$\frac{F_\Delta(x,n)}{n}.$$

Suppose on the orbit segment of a point x under the irrational rotation by α given by $\{R_\alpha^i(x)\}_{i=0}^m$, we found that given the arc Δ, that $R_\alpha^{k_1}(x), R_\alpha^{k_2}(x), R_\alpha^{k_3}(x) \subset \Delta$ and these were the only three. Then we know that the frequency $F_\Delta(x,m) = 3$, and the relative frequency $\frac{F_\Delta(x,m)}{m} = \frac{3}{m}$. In our example from above in the figures, the relative frequency of hits on the interval $N_\delta(0)$ on the orbit segment $\{R^i_\alpha(x)\}_{i=0}^{18}$ is $\frac{F_{N_\delta(0)}(0,18)}{18} = \frac{2}{18} = \frac{1}{9}$. The goal is to study the relative frequency of a rotation on any arc of any length and be able to say something meaningful about how often, on average, the *entire orbit* visits the arc.

Some notes:

- Define $\ell(\Delta)$ = length of Δ (under some metric).
- The relative frequency really does not depend on whether Δ is open, closed or neither (why not?).
- The convention is to take representatives for arcs to be of the "half-closed" form $[\cdot,\cdot)$. Then it is easy to see whether unions of arcs are connected or not.
- We study the overall relative frequency of entire orbits: This translates to a study of
$$\lim_{n\to\infty} \frac{F_\Delta(x,n)}{n}.$$
However, It is yet not entirely clear that this limit actually exists. We first address this point.

Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = \left(1 + \frac{1}{n}\right)\sin n$. A priori, we do not know whether $\lim_{n\to\infty} f(n)$ exists or not (Really, though, think of the continuous version of this function in calculus. There isn't a horizontal asymptote for f). So we first define the limit inferior (respectively superior) for f. This type of limit either always exists or is $-\infty$ (resp. ∞). It is the largest (resp. smallest) number where no more than a finite number of terms in the sequence are smaller (resp. larger) than it on the entire sequence. Think of the envelope of a sequence being defined to allow some terms to be outside the envelope, but only a finite number of them. In the case of $f(n) = \sin n$, the $\liminf_{n\to\infty} f(n) = -1$. This makes sense, since if we try to "cut" the function at anything above -1, that small interval of values (think $[-1, -1+\epsilon)$) will be visited an infinite number of times eventually by f. Also, $\limsup_{n\to\infty} f(n) = 1$. It should be obvious that while these quantities may not be easy to calculate, not only should they exist (for the minute, think of an infinite limit as existing in the sense that the sequence is going somewhere), but it must be the case that for any sequence $\{x_n\}_{n\in\mathbb{N}}$,
$$\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n.$$
And should they be equal, then $\lim_{n\to\infty} x_n$ in fact exists and is equal to the two limit bounds.

In our case, let A be a disjoint union of arcs. Then define
$$\overline{f}_x(A) = \limsup_{n\to\infty} \frac{F_A(x,n)}{n}, \quad \underline{f}_x(A) = \liminf_{n\to\infty} \frac{F_A(x,n)}{n}.$$
It turns out that these two quantities not only exist. They also are equal:

Proposition 7. *For any arc $\Delta \subset S^1$, and every $\overline{x} \in S^1$, and any irrational rotation R_α, $\alpha \notin \mathbb{Q}$ on S^1, we have*

$$f(\delta) := \lim_{n\to\infty} \frac{F_\Delta(x,n)}{n} = \ell(\Delta).$$

idea. The proof relies on finding bounds for the quantities $\overline{f}_x(\Delta)$ and $\underline{f}_x(\Delta)$, and showing that it is always the case that $\overline{f}_x(\Delta) \le \ell(\Delta)$ and $\underline{f}_x(\Delta) \ge \ell(\Delta)$. This can only be the case if the limits superior and inferior are in fact equal, and equal to $\ell(\Delta)$. □

Notes: Let $R_\alpha : S^1 \to S^1$ be an irrational rotation. Then for $\overline{x} \in S^1$,

- the orbit $\mathcal{O}_{\overline{x}}$, as a sequence $\{R_\alpha^n x\}_{n\in\mathbb{N}}$, is called a *uniform distribution* or an *equidistribution* on S^1.
- the orbit $\mathcal{O}_{\overline{x}}$ in a sense "fills" every arc in S^1.

Hence, we say that any orbit of an irrational rotation of S^1 is uniformly distributed on S^1. This is our notion of a set being dense in another set, and for these orbits, one can actually "see" the notion of recurrence. To further understand this new type of dynamical behavior, we will do an application next time. But first, we will start the next lecture with a little more nomenclature.