MATH 421 DYNAMICS

Week 5 Lecture 1 Notes

Linear Maps of $\mathbb{R}^2$ (cont'd.)

Last class we discussed the basic types of dynamic behavior that a linear map of $\mathbb{R}^2$ can exhibit. There were three basic cases. But notice that in Case I above, we missed a type. Today, we will explore this neglected type in detail.

Suppose A is a 2×2 matrix with eigenvalues $0 < |\mu| < 1 < |\lambda|$. Then $A \overset{\text{conj}}{\cong} B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ like the other examples in Case I, but the orbit lines are different. In fact, writing out the nth term in $\mathcal{O}_{\mathbf{v}}$ for a choice of $\mathbf{v} \in \mathbb{R}^2$, we see that there are four types:

(1) $\mathcal{O}^+_{\mathbf{v}} \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathcal{O}^-_{\mathbf{v}} \longrightarrow \infty$,
(2) $\mathcal{O}^+_{\mathbf{v}} \longrightarrow \infty$ and $\mathcal{O}^-_{\mathbf{v}} \longrightarrow \infty$,
(3) $\mathcal{O}^+_{\mathbf{v}} \longrightarrow \infty$ and $\mathcal{O}^-_{\mathbf{v}} \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and
(4) $\mathcal{O}^+_{\mathbf{v}} \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathcal{O}^-_{\mathbf{v}} \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

With B as our matrix, the eigenvectors $\mathbf{v}_\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ lie on the coordinate axes, and for a choice of $\mathbf{v} \in \mathbb{R}^2$, the nth term is agian $B^n\mathbf{v} = \begin{bmatrix} \lambda^n v_1 \\ \mu^n v_2 \end{bmatrix}$. Can you envision the orbit lines and motion along them? Do you recognize the phase portrait? Can you classify the type and stability of the origin?

Consider now the hyperbolic matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Here the characteristic equation is $r^2 - r - 1 = 0$, which is solved by $r = \frac{1 \pm \sqrt{5}}{2}$, giving us the eigenvalues

$$\lambda = \frac{1+\sqrt{5}}{2} > 1, \quad \text{and} \quad \mu = \frac{1-\sqrt{5}}{2} \in (-1, 0).$$

The eigenspace of λ is the line $y = \frac{1+\sqrt{5}}{2}x = \lambda x$, and for an eigenvector, we choose $\mathbf{v}_\lambda = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$.

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Now let $f : \mathbb{R}^2 \to \mathbb{R}^2$, be the linear map $f(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{v}$. Then, for $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we get

$$\mathcal{O} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 8 \\ 13 \end{bmatrix}, \dots \right\}.$$

Do you see the patterns? Call $\mathbf{v}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ and the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Fibonacci with $y_n = x_{n+1}$. Notice that the sequence of ratios

$$\left\{ \frac{y_n}{x_n} \right\}_{n\in\mathbb{N}} = \left\{ \frac{x_{n+1}}{x_n} \right\}_{n\in\mathbb{N}}$$

has a limit, and $\lim_{n\to\infty} \frac{y_n}{x_n} = \frac{1+\sqrt{5}}{2} = \lambda$.

Recall how to find this limit: Use the second-order recursion inherent in the Fibonacci sequence, namely $a_{n+1} = a_n + a_{n-1}$, and the ratio to calculate a first-order recursion. This first-order recursion will correspond to a map, which one can study dynamically. Indeed, Let $r_{n+1} = \frac{x_{n+1}}{x_n}$, Then

$$r_{n+1} = \frac{x_{n+1}}{x_n} = \frac{x_n + x_{n-1}}{x_n} = 1 + \frac{1}{\frac{x_n}{x_{n-1}}} = 1 + \frac{1}{r_n}.$$

So $r_{n+1} = f(r_n)$, where $f(x) = 1 - \frac{1}{x}$. The only non-negative fixed point of this map is the sole solution to $x = f(x) = 1 - \frac{1}{x}$, or $x^2 - x - 1 = 0$, which is $x = \frac{1+\sqrt{5}}{2}$. Note that really there are two solutions and the other one is indeed μ. However, since we are talking about populations, the negative root doesn't really apply to the problem.

Example 1. Recall the Lemmings problem, with its second-order recursion $a_{n+1} = 2a_n + 2a_{n-1}$. Here the sequence of ratios of successive terms $\left\{\frac{a_{n+1}}{a_n}\right\}_{n\in\mathbb{N}}$ has the limit $1 + \sqrt{3}$.

Here are two rhetorical questions:

(1) What is the meaning of these limits?
(2) How does the hyperbolic matrix in the above Fibonacci sequence example help in determining the limit?

To answer these, let's start with the sequence

$$\{b_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}.$$

As before, we see

$$\mathbf{v}_{n+1} = \begin{bmatrix} b_{n+1} \\ b_{n+2} \end{bmatrix} = \begin{bmatrix} b_{n+1} \\ b_{n+1} + b_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_n \\ b_{n+1} \end{bmatrix} = A \begin{bmatrix} b_n \\ b_{n+1} \end{bmatrix} = A\mathbf{v}_n,$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. This is precisely the matrix that (1) moves the second entry into the first entry slot, and (2) creates a new entry two by summing the two entries.

Here, we have associated to the second-order recursion $b_{n+2} = b_{n+1} + b_n$ the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and the first-order *vector* recursion $\mathbf{v}_{n+1} = A\mathbf{v}_n$.

Remark 2. This is basically a *reduction of order* technique, mush like the manner with which one would reduce a second-order ODE into a system of 2 first-order ODES, written as a single vector ODE.

This is actually used to construct a function which gives the nth term of a Fibonacci sequence in terms of n (rather than only in terms of the $(n-1)$st term):

Proposition 3.1.11. *Given the second order recursion $b_{n+2} = b_{n+1} + b_n$ with the initial data $b_0 = b_1 = 1$, we have*

$$b_n = \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda - \mu},$$

where $\lambda = \frac{1+\sqrt{5}}{2}$ and $\mu = \frac{1-\sqrt{5}}{2}$.

We showed that λ and μ were the eigenvalues of a matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and that the linear map on $\mathbb{R}^2$ given by A, $\mathbf{v}_{n+1} = A\mathbf{v}_n$, is in fact the first-order vector recursion for the second-order recursion in the proposition under the assignment $\mathbf{v}_n = \begin{bmatrix} b_n \\ b_{n+1} \end{bmatrix}$. This reduction-of-order technique for the study of recursions is quite similar to (and is the discrete version of) the technique of studying the solutions of a single, second-order, homogeneous, ODE with constant coefficients by instead studying the system of two first-order, linear, constant-coefficient, homogeneous ODEs. In fact, this analogy is much more robust, which we will see in a minute.

First, a couple of notes:

- For very large n,

$$b_n = \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda - \mu} \sim K\lambda^{n+1}.$$

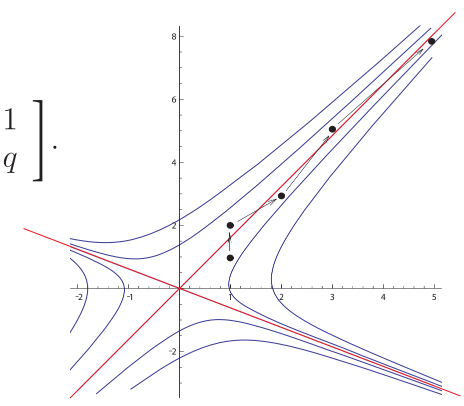
  Thus the growth rate of terms in the Fibonacci sequence are not exponential. They do, however, tend to look more and more exponential as n gets large. In fact, we can say the Fibonacci sequence displays *asymptotic exponential* growth, or that the sequence grows *asymptotically exponentially.*

- Start with the initial data $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and plot $\mathcal{O}_{\mathbf{v}_0}$ in the plane. What you will find is that the iterates of $\mathcal{O}^+_{\mathbf{v}}$ will live on two curves of motion (there will be flipping here across the $y = \lambda x$ eigenline. Why does this happen?) and tend toward the λ-eigenline as they grow off of the page (see the figure below). Getting closer to the λ-eigenline means that the growth rate is getting closer to the growth rate ON the λ-eigenline. But on this line, growth is purely exponential!. With growth factor $\lambda > 1$.

  Exercise 1. If we neglect the applcation of a rabbit population, the discrete dynamical system we constructed above is invertible. Calculate the first few pre-images of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and plot them on the figure below. Then calculate the orbit line equations for the orbit line on which the sequence lives. Hint: you may need to solve the original second-order ODE to do this.

- Every other point $\mathbf{v}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is really just another set of initial data for the second-order recursion (or the first-order vector version). Start taking iterates and plot and you will see that these orbits will also live on either one or will flip between two curves of motion and the phase diagram in the figure will tell you the ultimate fate of the orbits. Doing this, you should ask yourself the following questions:
  - **Q.** Can you find starting data which lead to a sequence which does NOT tend to run off of the page as n goes to infinity?
  - **Q.** If you can do the first, then can you do so in which the starting data are BOTH integers? Why or why not?

In general, let $a_{n+2} = pa_n + qa_{n+1}$ (careful of the order of the terms in this expression). Then we can construct a first-order vector recursion



$$\mathbf{v}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ p & q \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A\mathbf{v}_n, \text{ for } A = \begin{bmatrix} 0 & 1 \\ p & q \end{bmatrix}.$$

The characteristic equation of A is $r^2 - qr - p = 0$, with solutions $r = \frac{q \pm \sqrt{q^2+4p}}{2}$.

Proposition 3.1.13. *If* $\begin{bmatrix} 0 & 1 \\ p & q \end{bmatrix}$ *has two distinct eigenvalues* $\lambda \neq \mu$, *then every solution to the second-order recursion* $a_{n+2} = pa_n + qa_{n+1}$ *is of the form*

$$a_n = x\lambda^n + y\mu^n$$

where $x = \alpha v_1$ *and* $y = \beta w_1$*,* $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ *are respective eigenvectors of* λ *and* μ*, and* α *and* β *satisfy the vector equation*

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \alpha\mathbf{v} + \beta\mathbf{w}.$$

Remark 3. Hence the general second-order recursion and the first-order vector recursion carry the same information, and the latter provides all of the information necessary to completely understand the former. The method of solution is quickly discernable: Given a second-order recursion, calculate the data from the matrix A in the corresponding first-order vector recursion, including the eigenvalues and a pair of respective eigenvectors. Use this matrix data along with the initial data given with the original recursion to calculate the parameters in the functional expression for a_n.

Here is an example going back to our Fibonacci Rabbits Problem. Is essence, we use Proposition 3.1.13 to essentially prove Proposition 3.1.11.

Example 4. Go back to the original Fibonacci recursion $a_{n+2} = a_{n+1} + a_n$, with initial data $a_0 = a_1 = 1$. The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ has $\lambda = \frac{1+\sqrt{5}}{2}$ and $\mu = \frac{1-\sqrt{5}}{2}$ (as before) and using the notation of Proposition 3.1.13, one can calculate representative eigenvectors as $\mathbf{v} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ \mu \end{bmatrix}$. Thus $v_1 = w_1 = 1$. To calculate α and β, we have to solve the vector equation

$$\begin{aligned} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} &= \alpha\mathbf{v} + \beta\mathbf{w} \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \alpha\begin{bmatrix} 1 \\ \lambda \end{bmatrix} + \beta\begin{bmatrix} 1 \\ \mu \end{bmatrix} \end{aligned}$$

This is solved by $\alpha = \frac{1-\mu}{\lambda-\mu}$ and $\beta = \frac{\lambda-1}{\lambda-\mu}$ (verify this calculation!). Hence we have $x = \alpha v_1 = \frac{1-\mu}{\lambda-\mu}$ and $y = \beta w_1 = \frac{\lambda-1}{\lambda-\mu}$, and our formula for the nth term of the sequence is

$$a_n = \frac{(1-\mu)\lambda^n + (\lambda-1)\mu^n}{\lambda-\mu}.$$

This does not look like the form in Proposition 3.1.11, however. But consider that the term

$$(1-\mu) = \frac{2}{2} - \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} = \lambda,$$

and similarly $(\lambda - 1) = -\mu$, we wind up with

$$a_n = \frac{(1-\mu)\lambda^n + (\lambda-1)\mu^n}{\lambda-\mu} = \frac{\lambda\cdot\lambda^n + (-\mu)\cdot\mu^n}{\lambda-\mu} = \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda-\mu},$$

and we recover Proposition 3.1.11 precisely.

Exercise 2. Perform this calculation for the second-order recursion in the Lemmings Problem, and use it to calculate the population of lemmings today, given that the initial population was given in 1980.

Section 3.2 in the text is a very nice discussion of the relationship between the matrices found in first-order, 2-dimensional homogeneous linear systems (with constant coefficients) of ODEs and the corresponding matrices of the discrete, time-1 maps of those systems. This is good reading, and truly exposes a fact that is commonly confusing among new students to this discipline: Namely, why is it that for a ODE system with coefficient matrix A, the **sign** of the eigenvalues determines the stability of the equilibrium solution at the origin. But for a linear map of $\mathbb{R}^n$, it is the size of the **absolute values** of the eigenvalues that determine the stability of the fixed point at the origin. The matrix of the time-1 of an ODE system is NOT the same matrix as the coefficient matrix of the system. The two matrices are certainly related, but they are not identical. Furthermore, ANY ODE system has a time-1 map. But only certain types of linear maps correspond to the time-1 maps of ODE systems. It is really all about the exponential map. We will not develop this section in the class, but here are some facts. Let's start with an example:

Example 5. Calculate the time-1 map of the ODE system

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}.$$

This system is uncoupled and straightforward to solve. Using linear system theory, the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ are $\lambda = 2$ and $\mu = -1$, and, since A is diagonal, we can choose the vectors $\mathbf{v}_\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t},$$

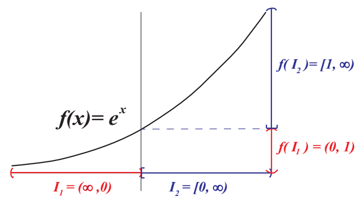
or $x_1(t) = c_1 e^{2t}$ and $x_2(t) = c_2 e^{-t}$. For the choice of any initial data, the particular solution is $x_1(t) = x_1^0 e^{2t}$ and $x_2(t) = x_2^0 e^{-t}$, and the evolution of this continuous dynamical system is

$$\varphi(\mathbf{x}, t) = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \mathbf{x}.$$

The time-1 map is then $\varphi(\mathbf{x}, 1) = \varphi_1(\mathbf{x}) : \mathbf{x}(0) \longmapsto \mathbf{x}(1)$, or the linear map

$$\varphi_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \varphi_1(\mathbf{x}) = B\mathbf{x},$$

where $B = \begin{bmatrix} e^2 & 0 \\ 0 & e^{-1} \end{bmatrix}$ is the matrix associated to the linear map.



Do you see the relationship between the ODE matrix A and the time-1 linear map matrix B. The type and stability of the equilibrium solution at the origin of this linear system given by A is that of a saddle, and unstable. The time-1 map must also be a saddle, as the orbit lines of the time-1 map coincide precisely to the solution curves of the ODE system. It is the sign of the eigenvalues (non-zero entries of A in this case) that determine the type and stability of the origin of the ODE system. However, it is the "size" (modulus) of the eigenvalues of B which determine the type and stability of the fixed point at the origin in the linear map given by B. Some notes:

- Notice that the *exponential map*, $\exp : x \mapsto e^x$ takes $\mathbb{R}$ to $\mathbb{R}_+$ (see the figure above) and maps all non-negative numbers, 0 and $\mathbb{R}_+$, to the interval $[1, \infty)$ and all negative numbers to $(0, 1)$. This is no accident, and exposes a much deeper meaning of the exponential map, which we will not go into here.
- One might conclude that there could not be a time-1 map of a linear, constant coefficient, homogeneous ODE system with negative eigenvalues. And you would be correct in this hyperbolic case. In general?
- One might also conclude that for any 2×2-matrix A, the associated time-1 map B would simply be the exponentials of each of the entries of A. Here, you must definitely be much more careful, as we shall see.

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map $f(\mathbf{x}) = B\mathbf{x}$, where $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and both $a > 0$ and $b > 0$. Determine a linear, 2-dimensional ODE system that has f as its time-1 map. For $a > 0$, show that B cannot correspond to a time-1 map of an ODE system if $b \leq 0$. Can B correspond to a time-1 map of an ODE system if both $a < 0$ and $b < 0$? Hint: The answer is yes.

For a moment, recall the 1-dimensional linear, homogeneous, constant coefficient ODE $\dot{x} = ax$, for $a \in \mathbb{R}$ a constant. The evolution is $x(t) = x_0 e^{at}$, the ODE is solved by an exponential function involving a. For the nth order linear, homogeneous, constant coefficient case, one creates an equivalent system $\dot{\mathbf{x}} = A\mathbf{x}$, a single vector ODE whose solution also seems exponential in nature (exponentials have the appeal that the derivative is proportional to the original function). That is, it is tempting to write the evolution as $\mathbf{x}(t) = \mathbf{x}^0 e^{At}$, since if it is the case that $\frac{d}{dt}\left[\mathbf{x}^0 e^{At}\right] = A\mathbf{x}^0 e^{At}$, then this expression solves the ODE. However, it is not yet clear what it means to take the exponential of a matrix.

Definition 6. For an $n \times n$ matrix A, define $e^A = \displaystyle\sum_{n=0}^{\infty} \frac{A^n}{n!}$.

This definition obviously comes directly from the standard definition of the exponential e^x via its Maclauren Series. Also, it seems to make sense in that one can certainly sum matrices, take them to positive integer powers, and divide them by scalars. The question of whether this converges or not is unclear, though. It really is a question of whether each entry, written as a series will converge. While this is basically a calculus question, we will not elaborate here but will state without proof the following: The definition above is well-defined and the series converges absolutely for all $n \times n$ matrices A.

Proposition 7. $\frac{d}{dt}\left[\mathbf{x}^0 e^{At}\right] = A\mathbf{x}^0 e^{At}$.

Proof. Really, this is just the definition of a derivative:

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \lim_{h\to 0}\frac{e^{A(t+h)} - e^{At}}{h} == \lim_{h\to 0}\frac{e^{At}e^{Ah} - e^{At}}{h} = e^{At}\lim_{h\to 0}\frac{e^{Ah} - 1}{h} \\
&= e^{At}\lim_{h\to 0}\frac{1}{h}\left(\sum_{n=0}^{\infty}\frac{(Ah)^n}{n!} - I\right) = e^{At}\lim_{h\to 0}\frac{1}{h}\sum_{n=1}^{\infty}\frac{(Ah)^n}{n!} \\
&= e^{At}\lim_{h\to 0}\sum_{n=1}^{\infty}\frac{A^n h^{n-1}}{n!} = e^{At}\lim_{h\to 0} A\sum_{n=1}^{\infty}\frac{A^{n-1}h^{n-1}}{n!} \\
&= Ae^{At}\lim_{h\to 0}\left(\frac{I}{1!} + \frac{Ah}{2!} + \frac{A^2h^2}{3!} + \frac{A^3h^3}{4!} + \dots\right).
\end{aligned}$$

At this point, every term in the remaining series has an h in it except for the $n = 1$ term, which is I. So

$$\frac{d}{dt}e^{At} = e^{At}\lim_{h\to 0} A\sum_{n=1}^{\infty}\frac{A^{n-1}h^{n-1}}{n!} = Ae^{At}.$$

□

Hence the expression e^{At} behaves a lot like the exponential of a scalar and in fact does solve the vector ODE $\dot{\mathbf{x}} = A\mathbf{x}$, with initial condition $\mathbf{x}(0) = \mathbf{x}^0$. However, contrary to Example 5, it is not in general true that the exponential of a matrix is simply the matrix of exponentials of the entries.

Example 8. Find the evolution for $\dot{\mathbf{x}} = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}\mathbf{x}$.

Here, the characteristic equation is $r^2 - r - 6 = 0$, with solutions giving eigenvalues of $\lambda = 3$ and $\mu = -2$. Calculating eigenvectors, we choose $\mathbf{v}_\lambda = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_\mu = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Thus the general solution is

$$\mathbf{x}(t) = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-2t} = \begin{bmatrix} 2e^{3t} & e^{-2t} \\ e^{3t} & 3e^{-2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{1}$$

Writing this in terms of $\mathbf{x}^0$ (in essence, finding the evolution), we get the linear system

$$\begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 + 3c_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Solving for c_1 and c_2 in terms of the initial conditions involves inverting the matrix, and $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$. Hence the evolution is

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} 2e^{3t} & e^{-2t} \\ e^{3t} & 3e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{3t} & e^{-2t} \\ e^{3t} & 3e^{-2t} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{6}{5}e^{3t} - \frac{1}{5}e^{-2t} & -\frac{2}{5}e^{3t} + \frac{2}{5}e^{-2t} \\ \frac{3}{5}e^{3t} - \frac{3}{5}e^{-2t} & -\frac{1}{5}e^{3t} + \frac{6}{5}e^{-2t} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}. \end{aligned}$$

Hence we can also say now that

$$e^{At} = \begin{bmatrix} \frac{6}{5}e^{3t} - \frac{1}{5}e^{-2t} & -\frac{2}{5}e^{3t} + \frac{2}{5}e^{-2t} \\ \frac{3}{5}e^{3t} - \frac{3}{5}e^{-2t} & -\frac{1}{5}e^{3t} + \frac{6}{5}e^{-2t} \end{bmatrix}, \quad \text{for} \quad A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$$

and that the time-1 map of this ODE is the linear map given by

$$e^{A} = \begin{bmatrix} \frac{6}{5}e^{3} - \frac{1}{5}e^{-2} & -\frac{2}{5}e^{3} + \frac{2}{5}e^{-2} \\ \frac{3}{5}e^{3} - \frac{3}{5}e^{-2} & -\frac{1}{5}e^{3} + \frac{6}{5}e^{-2} \end{bmatrix}.$$

SO how does one square these calculations into a general understanding of e^A? Via the properties of of a matrix exponential:

Proposition 9. *Let $A_{n\times n}$ be diagonalizeable. Then $A = SBS^{-1}$, where*

- *$B_{n\times n}$ is diagonal, and*
- *the columns of $S_{n\times n}$ form an eigenbasis of A.*

Proposition 10. *If $A_{n\times n}$ is diagonalizeable, then $e^A = Se^BS^{-1}$, where both B and e^B are diagonal.*

Proof. Note that since

$$e^A = \sum_{n=1}^{\infty} \frac{A^n}{n!} \quad \text{and} \quad \left(SAS^{-1}\right)^n = SA^nS^{-1},$$

we have

$$Se^BS^{-1} = S\left(\sum_{n=1}^{\infty} \frac{B^n}{n!}\right)S^{-1} = \sum_{n=1}^{\infty} \frac{SB^nS^{-1}}{n!} = \sum_{n=1}^{\infty} \frac{\left(SBS^{-1}\right)^n}{n!} = \sum_{n=1}^{\infty} \frac{A^n}{n!} = e^A.$$

$\square$

Example 11. Back to the previous system, with $\dot{\mathbf{x}} = A\mathbf{x}$, and $A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$. The general solution, written in Equation 1 was

$$\begin{aligned}
\mathbf{x}(t) &= \begin{bmatrix} 2e^{3t} & e^{-2t} \\ e^{3t} & 3e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} 2e^{3t} & e^{-2t} \\ e^{3t} & 3e^{-2t} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{6}{5}e^{3t} - \frac{1}{5}e^{-2t} & -\frac{2}{5}e^{3t} + \frac{2}{5}e^{-2t} \\ \frac{3}{5}e^{3t} - \frac{3}{5}e^{-2t} & -\frac{1}{5}e^{3t} + \frac{6}{5}e^{-2t} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = e^{At}\mathbf{x}^0.
\end{aligned}$$

But the middle equal sign in the last grouping can easily be written

$$\begin{aligned}
\mathbf{x}(t) &= \begin{bmatrix} 2e^{3t} & e^{-2t} \\ e^{3t} & 3e^{-2t} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} = Se^{Bt}S^{-1}\mathbf{x}^0,
\end{aligned}$$

where S is the matrix whose columns form an eigenbasis of A, and e^{Bt} is the exponential of the diagonal matrix B. Hence, as in the proposition, $e^{At} = Se^{Bt}S^{-1}$.

Exercise 4. Show that the time-1 map of the ODE system $\dot{\mathbf{x}} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\mathbf{x}$ is given by the linear map $f(\mathbf{x}) = B_1\mathbf{x}$, where $B_1 = \begin{bmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{bmatrix}$, but the time-$t$ map in general is NOT given by the linear map $B_t = \begin{bmatrix} e^{\lambda t} & e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$.

Exercise 5. Find the time-1 map of the IVP $\dot{\mathbf{x}} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}\mathbf{x}$, and use it to construct a form for the exponential of a matrix with purely imaginary eigenvalues.