

MATH 421 DYNAMICS

Week 4 Lecture 2 Notes

We begin today with one last example of relatively simple dynamics, albeit on a more complicated, rather fascinating set. We shall return to this set a few times in the future, as it is quite ubiquitous in dynamical systems theory. For now, we only introduce it as a way to provide yet another use for a contraction map.

1. A CANTOR SET

Simple dynamics also allows us to define some of the physical properties of the spaces that a map can “act” on. For example, consider the following subset of the unit interval $[0, 1]$:

Define $C_0 = [0, 1]$. On C_0 , remove the open middle third interval $(\frac{1}{3}, \frac{2}{3})$ and call the remainder C_1 . Hence $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Continue removing the middle third from each remaining closed interval from the previous C_n to construct C_{n+1} . Then define

$$C = \bigcap_{n=0}^{\infty} C_n.$$

C is called the Ternary Cantor Set. It has the following properties:

- There are no positive-length intervals in C .
- It is easy to see that many points in C can be written as a multiple of a power of $\frac{1}{3}$ (the end points of the intervals at each stage). It is harder to see there are many others also. One way to possibly see this is to use an alternate description of C as the set of points in the unit interval whose ternary expansion base 3 (like our decimal expansion but using only 0's 1's, and 2's) has no 1's. Note that here, the number $\frac{1}{3} = .100000\dots = .02222222\dots$, so $\frac{1}{3} \in C$.

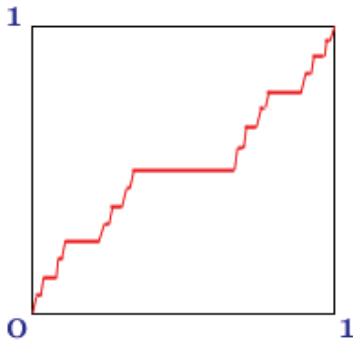
Exercise 1. Show in our standard decimal system, that $1.0 = .999999\dots$.

Why no 1's are allowed? Because we take them out. Think about this: In a ternary expansion of points in $[0, 1]$, the middle third (every point in the open interval $(\frac{1}{3}, \frac{2}{3})$) will have an expansion $.1xxxx\dots$ where not all of the x's are 0. But these points are not in C . Make sense?

- It is easy to see that there are infinitely many points in C . What is more interesting is that the number of points in C is uncountable (so not like the natural number kind of infinity. This is more like the number of points in $[0, 1]$ kind of infinity.) In fact, there are AS MANY points left in C as there were in the original $[0, 1]$! (Wrap your head around that mathematical fact!)
- One can give C a topology so that it is a space, like $[0, 1]$ is a space. Then one can say that anything homeomorphic to a Cantor Set is a Cantor Set. Hence this one example will share its properties with all other Cantor sets defined similarly.

- There is a wonderful, real-valued function defined on C which exposes some of its properties. This function, the Cantor-Lebesgue function, whose graph is sometimes called the Devil's Staircase (see Figure), is a continuous, function on C , surjective on $[0, 1]$, whose derivative is almost everywhere 0. Strange, eh? There is a picture on page 136.
- The map $f : C \rightarrow C$, $f(x) = \frac{x}{3}$ is a contraction whose sole fixed point is at 0.

This last point is special:



Definition 1. A set on which there exists a contraction map which is a homeomorphism onto its image has the property of *self-similarity*.

The set is also called re-scalable. This is easy to see for a contraction on $\mathbb{R}$.

Example 2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by the function $f(x) = \frac{2}{\pi} \arctan x$. Here, f is a homeomorphism onto its image which is $(0, 1)$. Viewed as a dynamical system, it is easy to see from the picture that iterating the map quickly leads to the conclusion that f is a contraction. What is the Lipschitz constant in this

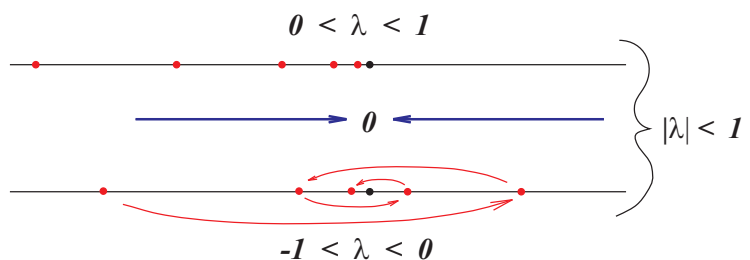
case? (hint: use the derivative!).

Exercise 2. In fact, every open interval $I = (a, b) \in \mathbb{R}$, for $a < b$, is homeomorphic to every other open interval in $\mathbb{R}$: Show that the map $f : I \rightarrow \mathbb{R}$, $f(x) = \frac{x - \frac{a+b}{2}}{(x-a)(b-x)}$ is a homeomorphism. Then show that, for $a = -1$, and $b = 1$, f is a contraction with unique fixed point $x = 0$.

Hence, $\mathbb{R}$ is a self-similar set. What is more interesting, though, are oddly defined sets which are self-similar. Examples include the Cantor set; If we started defining the Cantor Set using only one sub-interval at stage C_n , we would get exactly the same construction in the end, with exactly the same number of points and in their same relative positions, as the parent set. Else, use the contraction $f(x) = \frac{x}{3}$ as the homeomorphism from $[0, 1]$ to $[0, \frac{1}{3}]$. Either way, C is self-similar. Other examples include, the Koch Snowflake, the Sierpinski Gasket and Carpet, and the famous Mandelbrot Set. We will look at these in turn.

LINEAR MAPS OF $\mathbb{R}^2$

Recall for the moment the linear map of $\mathbb{R}$ defined by $f(x) = \lambda x$ (this can also be written $x \xrightarrow{f} \lambda x$). One can classify dynamical behavior of this map by the magnitude



of λ : Whether $|\lambda| < 1$, $|\lambda| = 1$, or $|\lambda| > 1$. Respectively, the origin is a sink and asymptotically stable, all points are fixed and all are stable but not asymptotically stable, or the origin is a source. However, the actual orbit structure is also affected by the sign of λ . Indeed, when $\lambda < 0$, the sequence becomes an alternating sequence and successive terms of the orbit will flip in sign (See the Figure). Dynamically, the orbits will tend to the same places. To see this, create a new dynamical system using the square of the map $f^2(x) = \lambda^2 x$. Then the new factor $\lambda^2 > 0$ always. The orbits of f^2 will stay on the side of the origin they started on. For f , however, the orbits will flip back and forth from one side of the origin to the other (see the figure). Hold this thought as we move on to linear maps of $\mathbb{R}^2$.

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map. Then for $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, we have $\mathbf{f}(\mathbf{v}) = A\mathbf{v}$, or

$$\begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- Here, $\mathbf{v}$ is an eigenvector of A , with eigenvalue λ if $\mathbf{v}$ satisfies the vector equation $A\mathbf{v} = \lambda\mathbf{v}$, or equivalently $(A - \lambda I)\mathbf{v} = \mathbf{0}$, where I is the 2-dimensional identity matrix.
- Recall the characteristic equation of A : $\det(A - \lambda I) = 0$, which can also be written

$$\lambda^2 - (\text{tr } A)\lambda + \det A = 0.$$

The roots are the eigenvalues of A .

A good questions to ask is: What information is conveyed by $\mathbf{v}$ and λ about the discrete dynamical system formed by iterating $\mathbf{f}$ on $\mathbb{R}^2$?

There is an easy classification of matrix types for A , and the classes are determined by the data of A :

- I. Two real distinct eigenvalues $\lambda \neq \mu$.
- II. One real eigenvalue λ . In this case, there are two possibilities:
 - $A = \lambda I$. Here A is called a *homothety* or a scaling.
 - A is conjugate to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. When $\lambda = 1$, this map is often called a *shear*.
- III. Two complex conjugate eigenvalues $\lambda = a + ib = \rho e^{i\theta}$, and $\mu = a - ib = \rho e^{-i\theta}$, where $\rho = a^2 + b^2$, and $\tan \theta = \frac{b}{a}$. Here, the effect of A is by rotation by θ and a scaling by ρ . When $\rho = 1$, the effect is a pure rotation (see below).

In actuality, there is a subclassification of these matrices which is of special interest to dynamicists. Every 2×2 matrix can be thought of as a scaled version of a determinant-1

matrix, since $\frac{1}{\det A}A = B$, where $\det B = 1$. Leaving aside the scale for a minute, we classify 2×2 matrices of determinant 1:

Let A be a 2×2 -matrix of determinant 1. Then we can classify A as:

- I. **Hyperbolic:** A has two real distinct eigenvalues $\lambda \neq \mu$ (necessarily $|\lambda| > 1 > |\mu|$).
- II. **Parabolic:** A has one real eigenvalue $\lambda = 1$ but is not diagonalizable. In this case, A is conjugate to $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ for some non-zero $s \in \mathbb{R}$.
- III. **Elliptic:** A has two complex conjugate eigenvalues $\lambda = e^{i\theta}$, and $\mu = e^{-i\theta}$.

Notice where the Identity Matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ shows up in this classification.

Exercise 3. Let A be a hyperbolic 2×2 matrix with integer coefficients. Show $|\operatorname{tr} A| > 2$ and λ is not rational.

Geometrically, choose a representative from each case above of determinant-1. Then it will be easier to see the effect on points in the plane. The figure is useful.

Proposition 3. *A linear map on $\mathbb{R}^n$ is eventually contracting iff all of its eigenvalues are of absolute value strictly less than 1.*

Note: We can define the *spectral radius* of a matrix A to be the quantity ρ_A , where

$$\rho_A = \left\{ \max_i |\lambda| \mid \lambda \text{ is an eigenvalue of } A \right\}.$$

Then we can easily restate the previous proposition:

Proposition 4. *A linear transformation A of $\mathbb{R}^n$ is eventually contracting if $\rho_A < 1$.*

There is a detailed discussion in the book in Section 3.3 on why it is not automatically the case that linear maps with all eigenvalues of modulus less than one are contracting. It involves the choice of norm and the data of the matrix. We will not spend time on this here, due to the time constraints of the course. However, in modeling, these kind of facts can be quite insightful.

Exercise 4. Construct an explicit example of a linear map on $\mathbb{R}^2$ which is eventually contracting but NOT a contraction.

Exercise 5. Show that no determinant-1 linear map on $\mathbb{R}^2$ can be eventually contracting. (Note: This has enormous implications in the mathematical models of physics and engineering, as it restricts the stability classification of both fixed points of maps and the equilibrium and periodic solutions of ODE systems.)

Some things to consider:

- Any linear map on $\mathbb{R}^2$ can be written as a linear combination of the three types in the figure (up to a scalar multiple, that is). Hence a detailed study of these three types is necessary to explore the dynamical structure of linear maps of the plane.
- Diagonalizing a matrix (conjugating it to one where the eigenvalues are prominent; This is really just a linear coordinate change) can be viewed as a change of the metric on $\mathbb{R}^2$. However, the new metric is always equivalent to the old one (you should show this!), where the definition of metric equivalence was given in the last lecture. It turns out that the process of diagonalization **does not** change the dynamical structure of the system!

Let $\mathbf{v} \in \mathbb{R}^2$ and consider $\mathcal{O}_{\mathbf{v}}$ under the linear map $f(\mathbf{v}) = A\mathbf{v}$. On iteration,

$$\mathbf{v} \mapsto A\mathbf{v} \mapsto A(A\mathbf{v}) = A^2\mathbf{v} \mapsto \dots \mapsto A^n\mathbf{v} \mapsto \dots$$

Hence, the orbit of $\mathbf{v}$ will depend critically on the data associated to A .

Case I. Suppose that the two eigenvalues of A are real and distinct, so that $\lambda \neq \mu$. Then there exists a matrix B , where $A \stackrel{\text{conj}}{\cong} B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$. Suppose further that $|\lambda| < |\mu| < 1$. Then, by the previous proposition, the origin is a sink and all orbits tend to $\mathbf{0}$. That is, $\forall \mathbf{v} \in \mathbb{R}^2, \mathcal{O}_{\mathbf{v}} \rightarrow \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. A deeper question is, however, how the orbit evolves as it tends to the origin.

For this, let's restrict the case further to the case where both eigenvalues are positive, so $0 < \lambda < \mu < 1$. Then, for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the n th term in the orbit sequence is

$$B^n \mathbf{v} = \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \mathbf{v} = \begin{bmatrix} \lambda^n v_1 \\ \mu^n v_2 \end{bmatrix}.$$

Observations:

- Since we have diagonalized the matrix A to get B , the n th term is easy to calculate as we have uncoupled the coordinates.
- The smaller eigenvalue λ means the first coordinate sequence in the orbit will have a faster decay toward 0 than that of the second coordinate. In the plane, this will imply a curved path toward the origin, with the orbit line (recall the definition of the

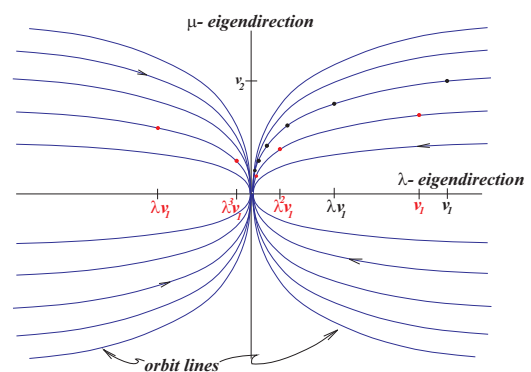


FIGURE 1. Phase Portrait for $f(\mathbf{v}) = B\mathbf{v}$, where $0 < \lambda < \mu < 1$.

orbit line?) bending toward the μ -eigendirection (why is this the case? Think about which coordinate is decaying faster.)

- Given the orbit lines in the figure, do you recognize the phase portrait? From the classification of 2-dimensional first-order, homogeneous linear systems (with constant coefficients), systems with this phase portrait have a node at the origin which is asymptotically stable. This is a sink.
- In the figure, the first few elements of $\mathcal{O}_{\mathbf{v}}$ are plotted in black.
- How does the phase portrait change if one or both of the eigenvalues of B are negative? As a hint, the orbit lines do not change at all. But the orbits, themselves? **In red in the figure are the first few elements of $\mathcal{O}_{\mathbf{v}}$ in the case that $0 < -\lambda < \mu < 1$.** Do you see the effect?

We can actually calculate the equations for the orbit lines: Let $x = v_1$ and $y = v_2$. Then the orbit lines satisfy the equation

$$|y| = C|x|^\alpha, \quad \text{where} \quad \alpha = \frac{\log |\mu|}{\log |\lambda|}.$$

Exercise 6. Derive this last equation for the orbit lines.

Exercise 7. The map above $f(\mathbf{v}) = B\mathbf{v}$, is the time-1 map of a first-order, linear homogeneous 2×2 system of ODEs. Find such a system and compare the matrix in the ODE system to B .

Exercise 8. Sketch the phase portrait in the case that $|\lambda| > |\mu| > 1$.

Case IIa. Suppose now that the linear map has a matrix with only 1 eigenvalue but 2 independent eigenvectors. That is, $A \stackrel{\text{conj}}{\cong} B = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \lambda I_2$, a homothety. Here, it should be clear that for any starting vector $\mathbf{v} \in \mathbb{R}^2$, the n th orbit element is $B^n \mathbf{v} = \lambda^n \mathbf{v}$. In the case that $0 < |\lambda| < 1$, we have $\mathcal{O}_{\mathbf{v}} \rightarrow \mathbf{0}$, for every $\mathbf{v} \in \mathbb{R}^2$. What does the motion look like in this case? Since we are simply re-scaling the initial vector $\mathbf{v}$, motion will be along the line through the origin given by $\mathbf{v}$. And the orbits will decay exponentially along these orbit lines. The phase portrait is that of a *star node* in this case, which is a sink, or implosion, when $|\lambda| < 1$, and a source, or explosion, when $|\lambda| > 1$. Again, think about what the orbits look like when $\lambda < 0$. Does anything change if the eigenvectors were $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$?

Case IIb. Suppose now that the linear map has a matrix with only 1 linearly independent eigenvector. In this case, $A \stackrel{\text{conj}}{\cong} B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Then $B^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} = \lambda^{n-1} \begin{bmatrix} \lambda & n \\ 0 & \lambda \end{bmatrix}$, and

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \xrightarrow{B^n} \lambda^{n-1} \begin{bmatrix} \lambda & n \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda^{n-1} \begin{bmatrix} \lambda v_1 + n v_2 \\ \lambda v_2 \end{bmatrix}.$$

Here, presence of the summand nv_2 has a twisting effect on γ_v even though the exponential factor λ^{n-1} still dominates the long-term orbit behavior.

Thus, if $|\lambda| < 1$, then $\forall \mathbf{v}$, $\mathcal{O}_v \rightarrow \mathbf{0}$, but the orbit lines twist (but do NOT rotate around the origin!)

This phase portrait exhibits a *degenerate node* and curves of motion (the orbit lines) are fairly complicated equations in the coordinates of the plane (see the book).

Case III. Suppose that the linear map has two complex conjugate eigenvalues

$$\lambda = \rho e^{i\theta} = a + ib, \quad \text{and} \quad \mu = \rho e^{-i\theta} = a - ib.$$

Then the orbit lines are

- spirals toward the origin if $0 < \rho < 1$,
- spirals away from the origin if $\rho > 1$, and
- Concentric circles if $\rho = 1$ (the eigenvalues then are purely imaginary).

Can you write the equations of the orbit lines in these cases?