

MATH 421 DYNAMICS

Week 4 Lecture 1 Notes

THE LOGISTIC MAP: LOW PARAMETER VALUES

We begin today with a look at the quadratic interval map called the logistic map: $f : [0, 1] \rightarrow [0, 1]$ given by $f(x) = \lambda x(1 - x)$. Last class, we showed that

Proposition 1. For $\lambda \in [0, 1]$, $\forall x \in [0, 1]$, $\mathcal{O}_x \rightarrow 0$.

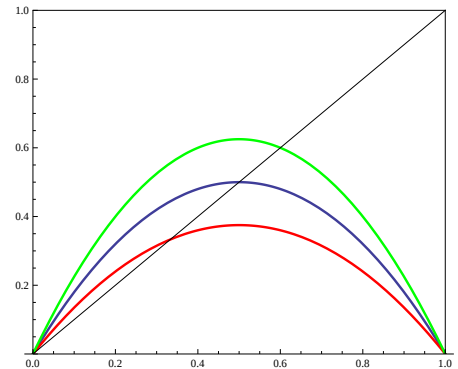
Today, we begin with the following:

Proposition 2. For $\lambda \in [1, 3]$, $\forall x \in (0, 1)$, $\mathcal{O}_x \rightarrow 1 - \frac{1}{\lambda}$.

Remark 3. if this is true, then $\lambda = 1$ is a bifurcation value for the family of maps f_λ , since

- for $\lambda \in [0, 1]$, the fixed point $x = 0$ is an attractor, and
- for $\lambda \in [1, 3]$, the fixed point $x = 0$ is a repeller (do you see this?).

The idea of the proof is that on this range of values for λ , the graph of f_λ intersects the line $y = x$ at two places, and these places are the two roots of $x = \lambda x(1 - x)$ (see proof of Proposition 1 from last class.) At right are the graphs for three typical logistic maps, for $\lambda = 1.5$, $\lambda = 2$, and $\lambda = 2.5$. It turns out that showing the fixed point $x_\lambda = 1 - \frac{1}{\lambda}$ is attractive is straightforward (this is an exercise). However, showing that almost every orbit converges to x_λ is somewhat more involved. I won't do the proof in class, as it is in the book. You should work through it to understand it well, since it raises some interesting questions. Like:



- (1) What generates the need for the two cases they describe in the book?
- (2) For what value(s) of λ is the attracting fixed point super-attracting?
- (3) The endpoints of the interval $\lambda \in [1, 3]$ are special and related in a very precise and interesting way. The property they share indicates a central property of attractive fixed and periodic points of C^1 -maps of the interval. Can you see this?

Once we surpass the value $\lambda = 3$ for $\lambda \in [0, 4]$, things get trickier. We will suspend our discussion of interval maps here for a bit and develop some more machinery first.

MORE GENERAL METRIC SPACES

There are easy-to-describe-and-visualize dynamical systems that occur on subsets of Euclidean space which are not Euclidean. As long as we have a metric on the space, it remains easy to discuss how points move around by their relative distances from each other. So let's generalize a bit and talk about metric spaces without regard to how they sit in a Euclidean space. To this end, let X be a metric space.

Definition 4. An ϵ -ball about a point $x \in X$ is the set

- (open) $B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$, and
- (closed) $\overline{B}_\epsilon(x) = \{y \in X \mid d(x, y) \leq \epsilon\}$.

Definition 5. A sequence $\{x_i\}_{i=1}^\infty \subset X$ is said to converge to $x_0 \in X$, if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall i \geq N$, $d(x_i, x_0) < \epsilon$.

Definition 6. A sequence is Cauchy if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall i, j \geq N$, $d(x_i, x_j) < \epsilon$.

Remark 7. A metric space X is called *complete* if every Cauchy sequence converges.

Definition 8. A map $f : X \rightarrow Y$ on a metric space X is called an *isometry* if

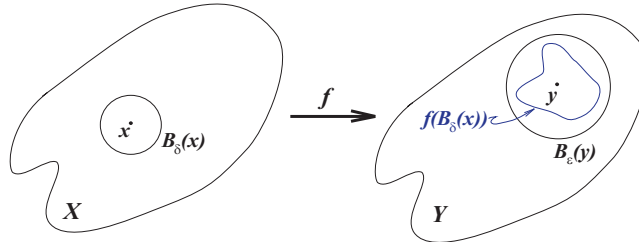
$$\forall x, y \in X, \quad d(f(x), f(y)) = d(x, y).$$

We can generalize this last definition to maps where the domain and the range are two different spaces: $f : X \rightarrow Y$, where both X and Y are metric spaces:

Definition 9. A map $f : X \rightarrow Y$ between two metric spaces X , with metric d_X , and Y , with metric d_Y , is called an *isometry* if

$$\forall x_1, x_2 \in X, \quad d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Definition 10. A map $f : X \rightarrow Y$ between two metric spaces is called *continuous* at $x \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$, such that if $\forall y \in X$ where $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \epsilon$.



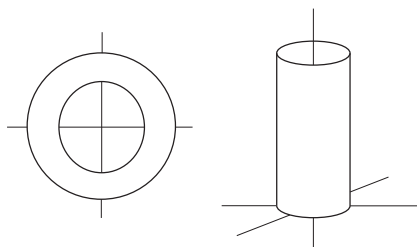
This gives us an easy way to define what makes a function continuous at a point when the spaces are not Euclidean. We will need this as we talk about common spaces in dynamical systems that are not like $\mathbb{R}^n$ but still allow metrics on them. That a map on any space is continuous is a vital property to possess if we are to be able to really talk at all about how orbits behave under iteration of the map. For now, though, more definitions:

Definition 11. A continuous bijection (remember that a bijection is a continuous map which is also an injection, or one-to-one map, as well as a surjection, or onto map) $f : X \rightarrow Y$ with a continuous inverse is called a *homeomorphism*.

Example 12. For any metric space (or any topological space in general!) X , the *identity map* on X ($f : X \rightarrow X$, $f(x) = x$) is a homeomorphism. It is obviously continuous (for any $\epsilon > 0$, choose $\delta = \epsilon$), one-to-one and onto, and it is its own inverse.

Example 13. Recall that linear maps $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = ax + b$ are, of course, invertible, as long as $a \neq 0$. However, be careful of the domain: Let $f(x) = \frac{1}{2}x + \frac{1}{2}$ on $I = [0, 1]$. Here, f is certainly injective (it is non-decreasing, and a contraction!). But it cannot have an inverse, since it is NOT onto I . In fact, the range of f is $[\frac{1}{2}, 1]$. So think about the following: For $f : I \rightarrow I$ to be a homeomorphism on a bounded $I = [a, b]$, it must be both one-to-one and onto. What does that imply about the images of the endpoints? Can you prove that there are only two possibilities for one, and once chosen, only one possibility for the other?

Remark 14. When a homeomorphism exists between two spaces, the two spaces are called *homeomorphic* and mathematically they are considered equivalent, or the same space. Anything defined on a space or with it can be defined or used on any other space that is homeomorphic to it. It is the chief way for mathematicians to classify spaces according to their properties.



Typically, on a metric space X , there are many metrics that one can define. However, like the above notion of homeomorphism, many of them are basically the same, and can be treated as equivalent. Others, maybe not. To understand this better,

Definition 15. Let d_1 and d_2 be two metrics on a metric space X . Then we say d_1 and d_2 are *isometric* if $\forall x, y \in X$, $d_1(x, y) = d_2(x, y)$.

Definition 16. Two metrics d_1 and d_2 on a metric space X are called (*uniformly*) *equivalent* if the identity map and its inverse are both Lipschitz continuous.

To elaborate on this last definition, we consider $f : X \rightarrow X$, $f(x) = x$, to be the map that takes points in X using the metric d_1 to points in x using the other metric d_2 . This is like considering X as two different metric spaces, one with d_1 and the other with d_2 . In essence, then the definition says that $\exists C, K \geq 0$ such that $\forall x, y \in X$, both 1) $d_2(f(x), f(y)) \leq$

$Cd_1(x, y)$ and 2) $d_1(f^{-1}(x), f^{-1}(y)) \leq Kd_2(x, y)$ hold. This simplifies using the identity map to 1) $d_2(x, y) \leq Cd_1(x, y)$ and 2) $d_1(x, y) \leq Kd_2(x, y)$ everywhere. Of course, what this really only means is that there are global bounds (over the space X , that is) on how the two metrics differ. We will see the utility of this later.

One last definition:

Definition 17. A map $f : X \rightarrow Y$ is called *eventually contracting* if $\exists C > 0$, such that if $\forall x, y \in X$ and $\forall n \in \mathbb{N}$,

$$d(f^n(x), f^n(y)) \leq C\lambda^n d(x, y)$$

for some $0 < \lambda < 1$.

There are many maps that are definitely not contractions, yet ultimately behave like one. Here is one:

Example 18. Let $f_2(x) = 2x(1-x)$ be the $\lambda = 2$ -logistic map, restricted to the open interval $(0, 1)$ (this cuts out the repelling fixed point at 0). This is the one with the super-attracting fixed point at $x = \frac{1}{2}$. Here f_2 is definitely NOT a contraction. You can see this visually by inspecting the graph: Should the graph of a function have a piece which is sloping up or down at a grade more than perfectly diagonal, the function will stretch intervals there. See the graph. To see this analytically, let $x = \frac{1}{8}$ and $y = \frac{1}{4}$. Then $f_2(x) = \frac{7}{32}$ and $f_2(y) = \frac{3}{8} = \frac{12}{32}$. Then

$$d(f_2(x), f_2(y)) = \left| \frac{12}{32} - \frac{7}{32} \right| = \frac{5}{32} \leq C \frac{4}{32} = C \frac{1}{8} = C \left| \frac{1}{4} - \frac{1}{8} \right| = Cd(x, y)$$

only when C is some number greater than 1. However, eventually, every orbit gets close to the only fixed point at $x = \frac{1}{2}$ where the derivative is very flat. The function f_2 , restricted to the interval $\left[\frac{3}{8}, \frac{5}{8}\right]$ is a $\frac{1}{2}$ -contraction (Can you show this? Use the derivative!). And one can also show that f_2 is 2-Lipschitz on all of $(0, 1)$. Thus, one can conclude here that f_2 is eventually contracting on $(0, 1)$, and $\forall x, y \in (0, 1)$,

$$d(f(x), f(y)) \leq 4 \left(\frac{1}{2} \right)^n d(x, y).$$

Exercise 1. Go back to the example $f(x) = \sqrt{x-1} + 3$ on the interval $I = [1, \infty)$. Show that f is NOT an eventual contraction (Hint: Try to find a value for C in Definition 17 that works in a neighborhood of $x = 1$.) Now show that f IS an eventual contraction on any closed interval $[b, \infty)$, for $1 < b < \frac{1}{2} + \frac{\sqrt{33}}{2}$.

Here are some of the more common non-Euclidean metric spaces encountered in dynamical systems:

(1) The n -dimensional sphere

$$S^n = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1 \right\}.$$

(2) The unit circle. Really this is the 1-dimensional sphere

$$S^1 = \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1 \right\}.$$

However, we also can use the interpretation of the plane as the 1-dimensional complex numbers

$$\begin{aligned} S^1 &= \left\{ z \in \mathbb{C} \mid |z| = 1 \right\}, \\ &= \left\{ e^{i\theta} \in \mathbb{C} \mid \theta \in [0, 2\pi) \right\}, \end{aligned}$$

and also in a more abstract sense as

$$S^1 = \left\{ x \in \mathbb{R} \mid x \in [0, 1] \text{ where } 0 = 1 \right\}.$$

This last definition requires a bit of explanation. From Set Theory, the concept of a partition of a set is a collection of exhaustive, mutually exclusive subsets. And since any space is really a set of its points with some additional structure, we have the following:

Definition 19. Given a set X , an *equivalence relation* R on X is a partition, where each element of the partition is called an equivalence class, and any two elements of each equivalence class are said to be equivalent or the same.

The notation for such an equivalence relation is the following: If $x, y \in X$ are in the same equivalence class, then we say $x \sim_R y$ (or simply $x \sim y$ when R is either obvious or implied), and for $x \in X$, we denote its equivalence class $[x]$. Thus, we can define the equivalence class of $x \in X$ as

$$[x] = \left\{ y \in X \mid y \sim x \right\}.$$

Furthermore, the set of all partition elements form a new set, called the *quotient set* of the equivalence relation. It is a deeper question exactly when the quotient set of an equivalence relation on a space is still a space. But for now, we say that the for X a set with an equivalence relation R , the quotient set is denoted $Y = X/R$.

Example 20. Any function $f : X \rightarrow \mathbb{R}$ defines an equivalence relation on X . Each element of the partition is simply the collection of all point that map to the same point in the range of X :

$$[x] = \left\{ y \in X \mid f(y) = f(x) \right\}.$$

Recall in Calculus, we defined the inverse image of a point in the range of a function as

$$f^{-1}(c) = \left\{ x \in X \mid f(x) = c \right\}.$$

Hence we can say that here that the equivalence class of a point $x \in X$ given by the function $f : X \rightarrow \mathbb{R}$ is simply the inverse image of the image of x , or $[x] = f^{-1}(f(x))$. This is well-defined regardless of whether f even has an inverse, since the inverse image of a function is only defined as a set. Think about this.

Using this last example, we have one more definition of S^1 . namely, let $r : \mathbb{R} \rightarrow S^1$ be a function $r(x) = e^{2\pi i x}$. Then $r(x) = r(y)$ iff $x - y \in \mathbb{Z}$ (work this out). In this case, each point on the circle has as its inverse image under r all of the points in the real line that are the same distance from the next highest integer (see the pic). Thus the map r looks like the real line $\mathbb{R}$ infinitely coiled around the circle. In this way, we commonly say that

$$S^1 = \mathbb{R}/\mathbb{Z}.$$

An interesting consequence of this idea? Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any function which is T -periodic (thus it satisfies $f(x + T) = f(x)$, $\forall x \in \mathbb{R}$). Then f induces a function $g : S^1 \rightarrow \mathbb{R}$ on S^1 , given by $g(t) = f(tT)$. Conversely, any function defined on S^1 may be viewed as a periodic function on $\mathbb{R}$, a tool that will prove very useful later on.

- (3) The cylinder: $C = S^1 \times I$, where $I \subset \mathbb{R}$ is some interval. Here I can be closed, open or half-closed, and can be bounded or all of $\mathbb{R}$. In fact, by the above discussion, any function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is T period in one of its variables, may be viewed as a function on a cylinder (think of the phase space of the undamped pendulum). Sometimes we call a cylinder whose linear variable is all of $\mathbb{R}$ the infinite cylinder.
- (4) The 2-torus $\mathbb{T}^2$ (or just the torus T when there is no confusion) $\mathbb{T} = S^1 \times S^1$. Like before, any function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is periodic in each of its variables, may be viewed as a function on a torus. Conversely, a function on the torus may be studied instead as a doubly periodic function on $\mathbb{R}^2$. We will use this also later.

We will continue next time with a brief look at Cantor sets and their properties.