MATH 421 DYNAMICS

Week 3 Lecture 2 Notes

Recall also that for any "nice" ODE in $\mathbb{R}^n$ (the definition of nice here is mathematical), in a neighborhood of an equilibrium solution, one can "linearize" the system. This means that, when possible, one can associate to this system a linear system whose equilibrium solution at the origin has the same properties as that of the original system, at least close by the equilibrium in study, Think of the tangent line approximation of a function at a point and you get the idea. Indeed, for the C^1-system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y), \end{aligned}$$

if (x_0, y_0) is an equilibrium solution, then the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is the linearized ODE system in a neighborhood of (x_0, y_0).

It is the main point of the celebrated Grobman-Hartman Theorem and the idea behind what is called *local linearization* that under certain conditions of the ODE system (that the vector field is C^1; this is the "nice" I mentioned before) and for certain classes of values of the eigenvalues of the linearized system, the origin of the linearized system, as an equilibrium solution, with be of the same type and have the same stability as that of (x_0, y_0). Keep this in mind.

Now, let $\mathbf{x} \in \mathbb{R}^n$ and $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be an ODE system. Define $\phi^t : \mathbb{R}^n \to \mathbb{R}^n$ the transformation of phase space given by the time-t map. Note that this is a "slice", for fixed t, of the flow in trajectory space given by $\phi(\mathbf{x}, t) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, so that

$$\phi_{t_0}(\mathbf{x}) = \phi(\mathbf{x}, t)\bigg|_{t=t_0}.$$

Let $\mathbf{p}$ be a T-periodic point which is NOT an equilibrium solution (think of the point $\left(1, \frac{\pi}{2}\right)$ in the $r\theta$-plane of the system above in the system from the last lecture $\dot{r} = r(1-r)$ and $\dot{\theta} = 1$.

Then $\mathbf{p} \in Fix(\phi^T)$, but $\mathbf{f}(\mathbf{p}) \neq \mathbf{0}$.

Lemma 1. 1 *is an eigenvalue of the matrix* $D\phi^T_{\mathbf{p}}$.

The proof is quite straightforward and really a vector calculus calculation. However, the implications are what is interesting here.

- For the time-T map which matches the period of the cycle perfectly, the point $\mathbf{p}$ appears as a fixed point (every point on the cycle will share this property.)

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- The time-T map $\phi^T : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation which takes the entire phase space to itself, and is in general non-linear.
- Since $\mathbf{p}$ is fixed, the derivative map $D\phi^T_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{R}^n \to T_{\mathbf{p}}\mathbb{R}^n$ is simply a linear transformation of the tangent space to $\mathbb{R}^n$ at the point $\mathbf{p}$.

Proof. The directional derivative of ϕ^T in the direction of the cycle (the curve parameterized by t, really) is the vector field $\mathbf{f}(\mathbf{p})$, and

$$\mathbf{f}(\mathbf{p}) = \mathbf{f}\left(\phi^T(\mathbf{p})\right) = \frac{d}{ds}\left(\phi^s\left(\mathbf{p}\right)\right)\Big|_{s=T} = \frac{d}{ds}\left[\phi^T \circ \phi^s\left(\mathbf{p}\right)\right]\Big|_{s=0} = D\phi^T_{\mathbf{p}}\left(\mathbf{f}(\mathbf{p})\right).$$

The first equality is because $\mathbf{p}$ is fixed, the second is due to the definition of a vector field given by an ODE, the third is because of the autonomous nature of the ODE and the last..., well, work it out. The end effect is that we have constructed the standard eigenvalue/eigenvector equation $\lambda\mathbf{v} = A\mathbf{v}$, where here $\lambda = 1$, $\mathbf{v} = \mathbf{f}(\mathbf{p})$ and the derivative matrix is A. □

Definition 2. For $\mathbf{p}$ a T-periodic point, call the other eigenvalues of $D\phi^T_{\mathbf{p}}$ the eigenvalues of $\mathbf{p}$.

Remark 3. The cycle (the periodic solution to the ODE system) becomes sort-of-like an equilibrium solution in many ways. It is another example of a closed, bounded solution who limits to itself. Solutions that start nearby may or may not stay nearby and may even converge to it. This give cycles the property of stability, much like equilibria. Many mechanical systems do exhibit asymptotically stable equilibrium states that are not characterized by the entire system staying still (think of the undamped pendulum, or more precisely a damped pendulum with just the right forcing function). How to analyze the neighboring solutions to see if a cycle is stable or not requires watching the evolution of these nearby solutions. The time-t map, and its cousin the First Return map, are ways to do this.

We end with a result that goes back to the notion of a contraction map:

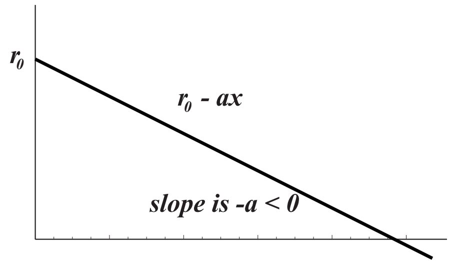
Proposition 4. *If* $\mathbf{p}$ *is a periodic point with all of its eigenvalues of absolute value strictly less than* 1*, then* $\mathcal{O}_{\mathbf{p}}$ *is an asymptotically stable limit cycle.*

Here the tangent linear map at $\mathbf{p}$ carries the infinitesimal variance in the vector field, which in turn betrays what the neighboring solutions do over time. The non-infinitesimal version theoretically plays the same role. Construct X, the closure of a small open subset of $\mathbb{R}^{n-1}$ centered at $\mathbf{p}$ and normal to the vector field $\mathbf{f}(\mathbf{p})$ at $\mathbf{p}$ (See the picture). Due to the continuity of the vector field given by the ODE system, all solutions of the ODE system that start in X sufficiently close to $\mathbf{p}$ will leave X, circle around, and again cross X. In the case where all solutions cross again at time-T (the period of $\mathbf{p}$), then the time-T map defines a discrete dynamical system on X. In the case where this is not the case (usual in nonlinear systems), then we neglect where the nearby solutions are at time T and simply look for where they again cross X. This latter case is the difference between a time-T map and the first return map. However, both of these constructions coalesce nicely into the infinitesimal version. We will revisit this point maybe later.

A Quadratic Interval Map: The Logistic Map

Like linear functions defined on the unit interval, discrete dynamical systems constructed via maps whose expression is a quadratic polynomial have many interesting properties. The ideal model for a study of quadratic maps of the interval is the Logistic Map. Before defining it, however, I want to motivate its prominence.

Consider the standard linear map on the real line, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = rx$. As a model for population growth (or decay), we restrict the domain to be non-negative (for a realistic population size) and the values for the parameter r to be positive, so that $f_r : [0, \infty) \to [0, \infty)$, where $r \geq 0$. Hence the recursive model is $x_{n+1} = f(x_n) = rx_n$, and again $\mathcal{O}_x = \{y \in [0, \infty) \,|\, y = f^n(x) = r^n x, n \in \mathbb{N}\}$. It is a good model for population growth when the population size is not affected by any environmental conditions or resource access, and is considered "ideal" growth. One way to view this is to say that in this case, "the growth factor r is constant and independent of the size of the population (see figure).



However, realistically speaking, unlimited population growth is unsustainable in any limited environment, and hence the actual growth factor winds up being dependent on the actual size of the population. Things like crowding and the finite allocation of resources typically mean that larger population sizes usually experience a dampened growth factor over time vis a vis small populations (think of a small number of fish in a large pond as opposed to a very large number of fish in the same pond). Hence a better model to simulate populations over time is to allow the growth factor to vary with the population size. The easiest way to do this is to replace the constant growth factor r, with one that varies linearly with population size. Here then replace r with the expression $r_0 - ax$, where r_0 is an ideal growth factor (for very small populations near 0), and a is a positive constant (see figure). The model becomes

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = (r_0 - ax)x,$$

or with a change in variables

$$f : \mathbb{R} \to \mathbb{R}, \quad f(y) = ky(1 - y).$$

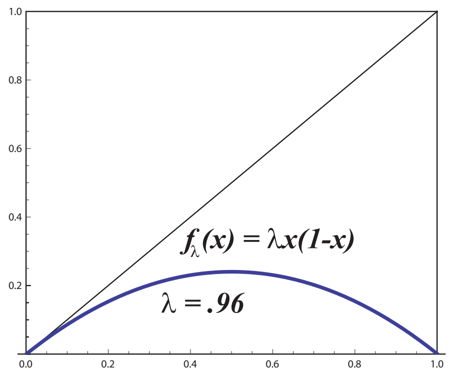
Keep in mind the limitations of the model as a guide to studying populations, however. For k a positive constant, f is positive only on the interval $[0, 1]$. And really only some values of k make this a good model for populations. To understand the last statement, you will need to actually see how k relates to the constants r_0 and a, and to study the graph of $r_0 - ax$ above as it relates to a population x.

Exercise 1. Do this change of variables.

Hence we will begin to study the dynamics of the map $f : [0,1] \to [0,1]$, $f(x) = \lambda x(1-x)$, called the logistic map. We will eventually see just how rich and complex the dynamics can actually be. For now, however, we will only spend time on the values of λ where the dynamics are simple to describe. First some general properties:

- f is only a map on the unit interval when $\lambda \in [0,4]$. Why does it fail for other values of λ?
- λ is sometimes called the *fertility constant* in population dynamics.
- We will use the notation f_λ to emphasize the dependence of f on the parameter.

Proposition 5. *For $\lambda \in [0,1]$, $\forall x \in [0,1)$, we have $\mathcal{O}_x \longrightarrow 0$.*



Visually, the graph of f_λ is a parabola opening down with horizontal intercepts at $x = 0, 1$. The vertex is at $\left(\frac{1}{2}, \frac{\lambda}{4}\right)$. And for $\lambda \in [0,1]$, the entire graph of f_λ lies below the diagonal $y = x$ (see figure at left). Cobweb to see where the orbits go.

Proof. The fixed points of f_λ satisfy $f_\lambda(x) = x$, or $\lambda x(1 - x) = x$. This is solved by either $x = 0$ or $x = \frac{\lambda - 1}{\lambda} = 1 - \frac{1}{\lambda}$. Hence for $0 \leq \lambda \leq 1$, the only fixed point on the interval $[0,1]$ is $x = 0$. Also, $\forall x \in [0,1]$, $f_\lambda(x) < x$. This implies that $\mathcal{O}_x$ is a decreasing sequence. As it is obviously bounded below, it must converge.

Now choose a particular $x \in [0,1]$ and notice that $f_\lambda(x) < \frac{1}{2}$. Thus, after one iteration of the map, every orbit lies inside of the subinterval $\left[0, \frac{1}{2}\right]$. So after one iteration, $f_\lambda|_{[0,1]}$ is a discrete dynamical system on a closed, bounded interval which is nondecreasing and has no fixed points on the interior $\left(0, \frac{1}{2}\right)$. Then by the proposition in class (Lemma 2.3.2), the end point 0 is fixed and all orbits converge to it. $\square$

Some notes:

- Both conditions, that the interval be closed, and that the map be nondecreasing, are necessary to apply Lemma 2.3.2. Since the original map f_λ was not nondecreasing, and the interval was open at 1, we needed to modify the situation a bit to fit the lemma. The nice structure of the graph of f_λ allowed for this by looking for a future iterate where the map would be nondecreasing. This is a common idea, and the basis for the notion of a map being *eventually nondecreasing*. Look for this in other maps in this class and beyond.

- The orbit $\mathcal{O}_1$ is special:
$$\mathcal{O}_1 = \{1, 0, 0, 0, \ldots\}.$$
The point $x = 1$ is called a pre-image of the fixed point $x = 0$. This is often seen in maps which are not one-to-one. The orbit $\mathcal{O}_1$ is called *eventually fixed*. There also exist *eventually periodic* points also. Both of these can not exist in invertible maps (why?), but it is easy to see that the quadratic map f_λ is NOT invertible on $[0, 1]$. But for now, realize that Proposition 5 is actually valid for $x \in [0, 1]$, including $x = 1$. I left it out originally due to its special nature.
- Were this logistic map with this range of λ to be used to model populations, one can conclude immediately the following:

All starting populations are doomed! Think about that.