MATH 421 DYNAMICS

Week 3 Lecture 1 Notes

We begin today's lecture with a restatement of our last assertion from last class:

Proposition 1. *Let the C^0-map $f : [\alpha, \beta] \to [\alpha, \beta]$ be non-decreasing, and suppose there are no fixed points on (α, β). Then either*

- *exactly one end point is fixed and $\forall x \in [\alpha, \beta]$, $\mathcal{O}_x$ converges to the fixed end point, or*
- *Both end points are fixed, one is an attractor and the other is a repeller.*

And if in the second case above, f is also invertible (f increasing is enough for this to be the case), then $forall x \in (\alpha, \beta)$, $\mathcal{O}_x$ is forward asymptotic to one end point, and backward asymptotic to the other.

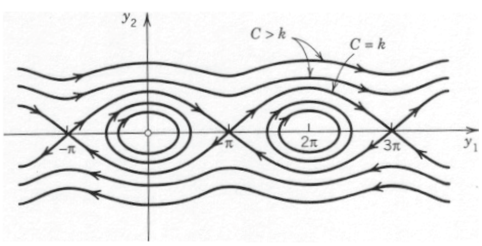
And using the notation for the forwards and backwards orbits I introduced from the last class, we can close in on a property of nondecreasing interval maps. First, we can now say definitively that

- In the first case, $\forall x \in [\alpha, \beta]$, either $\mathcal{O}^+ \longrightarrow \alpha$ or $\mathcal{O}^+ \longrightarrow \beta$. Here, f will certainly look like a contraction. Must it be? Keep in mind that the definition of a contraction is very precise, and maps that behave like contractions may not actually be contractions. Think $f(x) = x^2$ on the closed interval $[0, .6]$. What is $\text{Lip}(f)$ here?
- For f in the second case, and with f invertible, then $\forall x \in (\alpha, \beta)$, either $\mathcal{O}_x^+ \longrightarrow \alpha$ and $\mathcal{O}_x^- \longrightarrow \beta$, or $\mathcal{O}_x^+ \longrightarrow \beta$ and $\mathcal{O}_x^- \longrightarrow \alpha$.

Let's elaborate on this forward/backward orbit thing. Suppose now that $f : X \to X$ is a C^0-map on some subset of $\mathbb{R}^n$, and suppose $\exists x \in X$, where

$$\mathcal{O}_x^- \longrightarrow a \text{ and } \mathcal{O}_x^+ \longrightarrow b.$$

Definition 2. x is said to be heteroclinic to a and b if $a \neq b$, and homoclinic to a if $a = b$.



You have certainly seen heteroclinic and homoclinic orbits before. Think of the phase plane of the undamped pendulum. It is the famous picture at left. Here the separatrices are heteroclinic orbits from the unstable equilibrium solution at $(2n\pi, 0)$, $n \in \mathbb{Z}$, and $(2(n \pm 1)\pi, 0)$. Although, in reality, there is a much more accurate picture of the phase space of the undamped pendulum. The vertical variable (representing the instantaneous velocity of the pendulum ball, actually) takes values in $\mathbb{R}$, while the horizontal variable (representing the angular position of the pendulum

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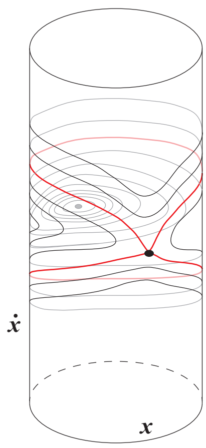
ball with respect to downward vertical position) is in reality 2π-periodic. Truly, it takes values in the circle S^1:

$$S^1 = \text{ unit circle in } \mathbb{R}^2 = \left\{ e^{i\theta} \in \mathbb{C} \,\middle|\, \theta \in [0,1) \right\}.$$

Thus, the phase space is really a cylinder, and really only has two equilibrium solutions; one at $(0,0)$, and the other at $(\pi,0)$. In this view, which we will elaborate on later, there are only two separatrices (both in red at right), and both are homoclinic to the unstable equilibrium at $(\pi,0)$. Also, it becomes clear once the picture is understood that ALL orbits of the undamped pendulum, except for the separatrices, are periodic. However, the period of these orbits is certainly not all the same. And there is NO bound to how long a period may actually be. See if you can fully grasp this.



Exercise 1. Show that there cannot exist homoclinic points for f a nondecreasing map on a closed, bounded interval.

Exercise 2. Construct an example (with an explicit expression) of a continuous C^0-map of S^1 that contains a homoclinic point. Now construct a C^1-map on S^1 with this property. (Hint: In class we already have an example of an interval map that, when modified, will satisfy the C^0 construction.

It turns out, forcing a map of an interval to be nondecreasing and forcing the interval to be closed really restricts the types of dynamics that can happen. We have:

Proposition 3. *Let $f : [\alpha,\beta] \to [\alpha,\beta]$ be C^0 and nondecreasing. Then $\forall x \in [\alpha,\beta]$, either x is fixed, or asymptotic to a fixed point. And if f is increasing (and thus invertible), then $\forall x \in [\alpha,\beta]$, either x is fixed or heteroclinic to adjacent fixed points.*

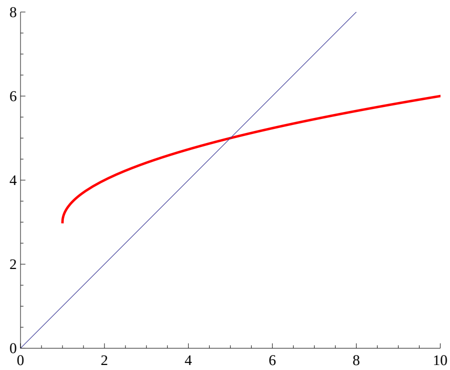
Clearly, the dynamics, although more complicated than for contraction maps, are nonetheless rather simple for nondecreasing interval maps (and even more so for invertible interval maps). Thus goes the second stop in our exploration of dynamical systems from simple to complex.

Before moving on, here are a few other things to think about: First, in the case of nondecreasing maps, the orbits of points in between fixed points are trapped in the interval bounded by the fixed points (draw some pictures to see this). This has enormous implications, and severely restricts what orbits can do (we say it restricts the "orbit structure" of the map). It makes the above proposition at the beginning of the lecture much more important and consequential, since any nondecreasing interval map is now simply a collection of disjoint interval maps, each of one of the types in the proposition. And also, it helps to establish the types of fixed points one can have for an interval map.

Exercise 3. Let f be a nondecreasing map on a closed interval. Show if $x_0 \neq y_0$ are two fixed points of f, then $\forall x \in (x_0, y_0)$, $\mathcal{O}_x^+ \subset (x_0, y_0)$.

Exercise 4. Let f be a C^0 nondecreasing map on a closed interval. Show that for every $n \in \mathbb{N}$, $Per_n(f) = Fix(f)$. That is, there are no non nontrivial periodic points of nondecreasing interval maps.

Example 4. For $f(x) = \sqrt{x-1} + 3$ on $I = [1, \infty)$, determine the set $Per(f)$.



We can address this question in a number of ways. First, notice that f is a strictly increasing function since $f'(x) = \frac{1}{2\sqrt{x}} > 0$ on all of I. Hence, every orbit is monotonic (not necessarily strictly, though. Why is this true?) Hence, as a generalization of the conditions of Proposition 3, every point is either fixed or has an orbit asymptotic to a fixed point, OR is unbounded (the consequence of an unbounded domain). At this point, you can conclude that there are no non-trivial periodic points, like in the above exercise. Now since the derivative is a strictly decreasing function, if there is a fixed point at all, it will be unique (think about this also). There is a fixed point of this map. To see this, consider the related differentiable function $g(x) = f(x) - x$. Here, $g(1) = 2 > 0$, while $g(10) = -4 < 0$, so by the Intermediate Value Theorem, there exists a pt $x_0 \in (1, 10)$, where $g(x_0) = 0$, or where $x_0 = f(x_0)$. Also, since $f(x) > x$ on $[1, x_0)$, and $f(x) < x$ on (x_0, ∞), the fixed point x_0 is an attractor. Hence $Per(f) = \{x_0\} = \left\{\frac{1}{2} + \frac{\sqrt{33}}{2}\right\}$, solving $f(x) = x$ algebraically.

Perhaps an easier way: $f(x)$ is certainly NOT a contraction, as for all $x \in [1, 2]$, the distance between images is actually greater than the original distances (verify this!). However, consider that the image of I, $f(I) = [3, \infty)$, and restricted to $f(I)$, f is actually a λ-contraction, with $\lambda = 3 + \sqrt{2}$. Hence, one could simply start iterating after the first iterate, knowing that the long-term behavior of orbits, fixed and periodic points, convergent orbits and stability, will all be the same. Thus, the map $f : f(I) \to f(I)$ is a contraction, and hence will have a unique fixed point and no other periodic points. Hence the x_0 found above is precisely all of $Per(f)$.

Remark 5. It would be tempting to call the map f above map *eventually contracting*, since it is a contraction on a forward iterate of the domain. However, this is not the case here. As we will see in soon enough, there is a technical condition that makes a map an eventual contraction, and there is a pathology at $x = 1$ here (pay attention to the derivative as one approaches 1 from numbers larger than 1). Suffice it to say that the language is not completely settled here.

Now take an increasing interval map f and vary it slightly. Usually, the dynamical behavior of the "perturbed" map stays the same (the number and type of fixed points does not change, even thought their position may very a bit). This may not be the case for a non-decreasing map: A slightly perturbed increasing map will remain increasing, while a slightly perturbed map with a flat interval may not remain nondecreasing. Think about that.

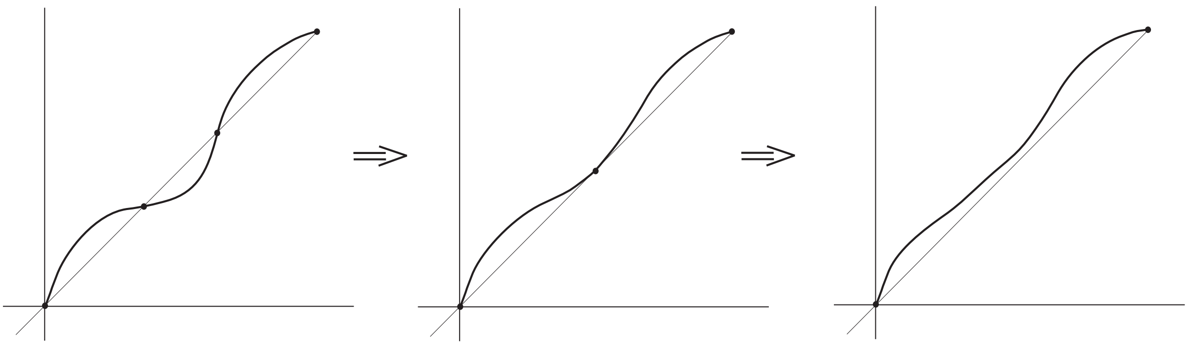
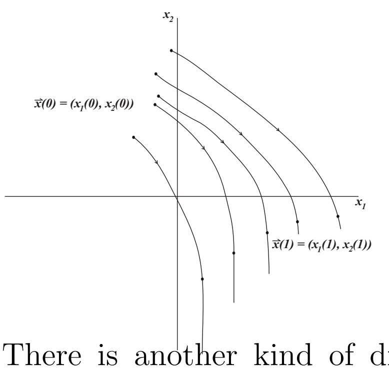


FIGURE 1. A bifurcation in an interval map.

And sometimes, a small change in an increasing map may lead to a big change is the number and type of fixed points (i.e., a big dynamical change!). Consider the three graphs below. Do you recognize this behavior? Have you ever seen a bifurcation in a mechanical system with a parameter?

1. First Return Maps

We will skip the sections on phase lines and 1-dimensional ODEs due to their full coverage in the prerequisite course for this one: 110.302 Differential Equations. Instead, we will jump directly to Section 2.4.3 in the book.

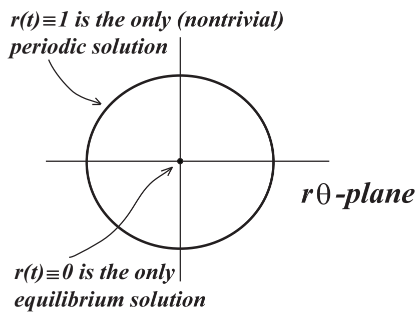


Recall that the time-1 map of an ordinary differential equation defines a discrete dynamical system on the phase space. Indeed, for $\mathbf{x} \in \mathbb{R}^n$, the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ defines the map

$$\phi_1 : \mathbb{R}^n \to \mathbb{R}^n, \quad \phi_1 : \mathbf{x}(0) \mapsto \mathbf{x}(1)$$

which is a transformation of $\mathbb{R}^n$. Really, $t = 1$ is only one such example, and any t will work, so long as the system solutions are defined (and unique, for the most part).

There is another kind of discrete dynamical system that comes from a continuous one: the First Return Map. One can view the first return map as a local version (only

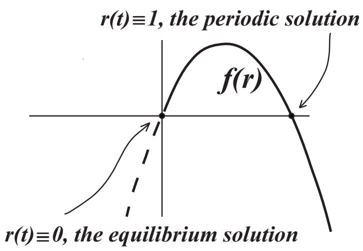


defined near interesting orbits) of the more globally defined time-t map (defined over all of phase space). Let's start with a 2-dimensional version. Consider the system on polar coordinates:

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= 1 \end{aligned} . \tag{1}$$

Without solving this system (although this is not difficult as the equations are uncoupled), we can say a lot about how solutions behave:

- The system is autonomous, so when you start does not matter, and the vector field is constant over time,
- The only equilibrium solution is at the origin. The second equation in the system really states that no point is fixed when θ is uniquely defined (on $[0, 2\pi)$, that is) for a choice of point in the plane. But the origin is special in polar coordinates.
- $r(t) \equiv 1$ is a periodic solution (the only one?) and called a *cycle*. What is the period?
- $r(t) \equiv 1$ is asymptotically stable as a cycle, and is called a *limit cycle*. Can you see why?

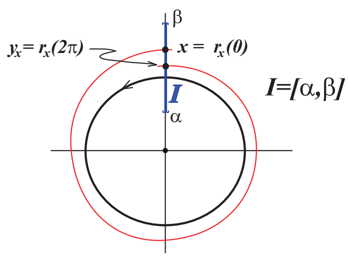


Now define

$$I = \left\{ [\alpha, \beta] \subset \text{ vertical axis } \middle| 0 < \alpha < 1, \beta > 1 \right\}.$$

For each $x \in I$, Call $r_x(t)$ the solution of Equation 1 passing through x at $t = 0$, so that $x = r_x(0)$. Let y_x be the point in I which corresponds to the earliest positive time that the resulting $r_x(t)$ again crosses I. (1) It must cross again (why?), and (2), really $y_x = r_x(2\pi)$. Then the map $\phi : x \mapsto y_x$ defines a discrete dynamical system on I.

Some properties of this discrete dynamical system should be clear:



- The dynamics are simple on I: There is a unique fixed point at $x = 1$ corresponding to the limit cycle crossing. This fixed point is asymptotically stable so that $\forall x \in I$, $\mathcal{O}_x \longrightarrow 1$. Thus this discrete dynamical system behaves like a contraction on I.
- The same can be said for the system

$$\begin{aligned} \dot{r} &= g(r) = r\left(\tfrac{1}{2} - r\right)(r-1)\left(\tfrac{3}{2} - r\right) \\ \dot{\theta} &= -1 \end{aligned} , \tag{2}$$

but only if I is chosen more carefully: Here $I = \left\{ [\alpha, \beta] \subset \text{ vertical axis } \middle| \tfrac{1}{2} < \alpha < 1 < \beta < \tfrac{3}{2} \right\}$.

- You should draw pictures to verify this. In this last system, what happens near the cycles $r(t) \equiv \frac{1}{2}$ and $r(t) \equiv \frac{3}{2}$? Is there some kind of discrete dynamical system in the form of a first return map near there also?