110.421 DYNAMICAL SYSTEMS

Week 1 Lecture 2 Notes

1. Some Simple Dynamics

To motivate our first discussion and set the playing field for a discussion of some simple dynamical systems, let's recall some general theory of first-order autonomous ODEs in one dimension (we call such an ODE, along with its initial value, an Initial Value Problem (IVP): Let

$$\dot{x} = f(x), \quad x(0) = x_0$$

be such an IVP where the function $f(x)$ is a differentiable function on all of $\mathbb{R}$. From any standard course in differential equations, this means that solutions will exist and be uniquely defined for all values of $t \in \mathbb{R}$ near $t = 0$ and for all values of $x_0 \in \mathbb{R}$. Recall that the general solution of this ODE will be a 1-parameter family of functions $x(t)$ parameterized by x_0. In reality, one would first use some sort of integrate technique (as best as one can; remember this ODE is always separable, although $\frac{1}{f(x)}$ may not be easy to integrate) to find $x(t)$ parameterized by some constant of integration C. Then one would solve for the value of C given a value of x_0. Indeed, one could solve generally for C as a function of x_0, and then substitute this into the general solution, to get

$$x(t, x_0) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

as the evolution. Then, for each choice of x_0, we would get a function $x_{x_0}(t) : \mathbb{R} \to \mathbb{R}$ as the particular solution to the IVP. We will use the notation with a subscript for x_0 to accentuate that the role of x_0 is that of a parameter. Specifying a value means solving the IVP for that value of x_0. Leaving x_0 unspecified means that we are looking for a particular solution at a fixed value of x_0. The resulting graph of $x_{x_0}(t)$ would "live" in the tx-plane as a curve (the trajectory) passing through the point $(x_0, 0)$. Graphing a bunch of representative trajectories gives a good idea of what the evolution looks like. You did this in your differential equations course when you created phase portraits.

Example 1. Let $\dot{x} = kx$, with $k \in \mathbb{R}$ a constant. Here, a general solution to the ODE is given by $x(t) = Ce^{kt}$. If, instead, we were given the IVP $\dot{x} = kx$, $x(0) = x_0$, the particular solution would be $x(t) = x_0 e^{kt}$. The trajectories would look like graphs of standard exponential functions (as long as $k \neq 0$) in the tx-plane. Below in Figure 1 are the three cases which look substantially different from each other: When $k > 0$, $k = 0$, and $k > 0$.

Recall in higher dimensions, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, we typically do not graph solutions explicitly as functions of t. Rather, we use the t-parameterization of solutions $\mathbf{x}(t) = [x_1(t) \;\; x_2(t) \;\; \cdots \;\; x_n(t)]^T$ to trace out a curve directly in the $\mathbf{x}$-space. This space, whose coordinates are the set of dependent variables $x_1, x_2, \ldots, x_n$, is called the phase-space (sometimes the tx-plane from

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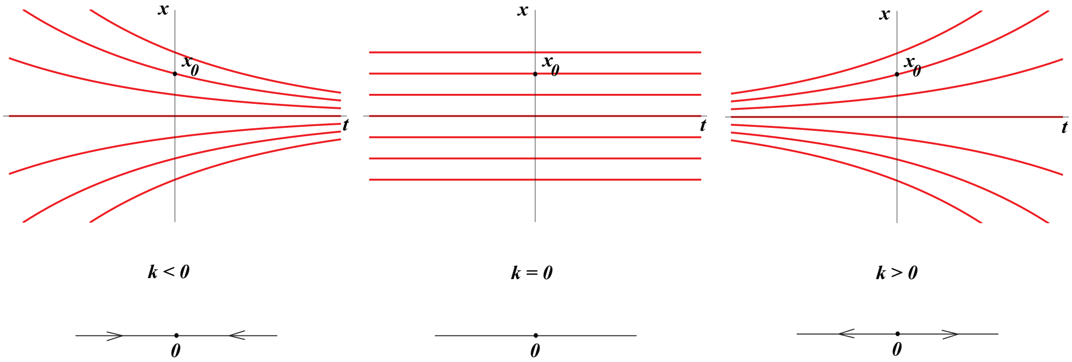

FIGURE 1. Sample solutions for $x_{x_0}(t) = x_0 e^{kt}$.

above, or more generally the $t\mathbf{x}$-space is called the *trajectory space* to mark the distinction). The diagrams in the plane that correspond to linear systems with a saddle at the origin, or a spiral sink are examples of phase planes with representative trajectories. Often, particularly in phase space, trajectories are also called *orbits*.

Even for autonomous ODEs in one-dependent variable, we played with a schematic diagram called a phase-line to give a qualitative description of the "motion" of solutions to $\dot{x} = f(x)$.

Example 2. The phase lines for $\dot{x} = kx$ for the three cases in the figure are below the graphs. The proper way to think of these lines is as simply a copy of the vertical axis (the x-axis in this case) in each of the graphs, marking the equilibrium solutions as special points, and indicating the direction of the x-variable change as t increases. All relevant information about the long-term behavior is encoded in these phase lines. In fact, these lines ARE the 1-dimensional phase spaces of the ODE, and the arrows simply indicate the direction of the parameterized $x(t)$ inside the line. It is hard to actually see the parameterized curves, since they all run over the top of each other. This is why we graph solutions in 1-variable ODEs using t explicitly, while for ODEs in two or more dependent variables, we graph using t implicitly, as the coordinate directly ON the curve in the phase space.

Again, for $\dot{x} = f(x)$, $x(0) = x_0$, the general solution $x(t, x_0) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a 1-parameter family of solutions, written as $x_{x_0}(t)$, parameterized by x_0. However, we can also think of this family of curves in a much more powerful way: As a 1-parameter family of transformations of the phase space! To see this, rewrite the general solution as $\varphi(t, x_0) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ instead of the possibly confusing notation $x(t, x_0)$. Now instead of thinking of x_0 as the parameter, fixing the second argument and varying the first as the independent variable, do it the other way: Fix a value of t, and allow the variable $x_0 = x$ (the starting point) to vary. Then we get for $t = t_0$:

$$\varphi(t_0, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \varphi_{t_0}(x) : \mathbb{R} \to \mathbb{R}, \quad x(0) \longrightarrow x(t_0).$$

As t varies, every point $x \in \mathbb{R}$ (thought of as the initial point $x(0)$, gets "mapped" to its new position at $x(t_0)$. Since all solutions are uniquely defined, this is a function for each value of t_0, and will have some very nice properties. But this alternate way of looking at the solutions of an ODE, as a family of transformations of its phase space, is the true *dynamical view*, and one we will explore frequently.

Definition 3. For $f : X \to X$ a map, define the set

$$\mathcal{O}_x = \left\{ y \in X \mid y = f^n(x), \ \ n \in \mathbb{N} \right\}$$

as the (forward) orbit of $x \in X$ under f.

Some notes:

- If f is invertible, we can define the backward orbit for $n \in -\mathbb{N}$, or the full orbit for $n \in \mathbb{Z}$.
- We can also write $\mathcal{O}_x = \left\{ x, f(x), f^2(x), \dots \right\}$, or for $x_{n+1} = f(x_n)$, $\mathcal{O}_x = \left\{ x_0, x_1, x_2, \dots \right\}$.

Consider the discrete dynamical system $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = rx$, $r > 0$. What do the orbits look like? Basically, for $x \in \mathbb{R}$, we get

$$\mathcal{O}_x = \left\{ x, rx, r^2x, r^3x, \dots, r^nx, \dots \right\}.$$

In fact, we can "solve" this dynamical system by constructing the evolution

$$\Phi(x, n) = r^n x.$$

Do the orbits change in nature as one varies the value of r? How about when r is allowed to be negative? How does this relate to the ordinary differential equation $\dot{x} = kx$?

Definition 4. For $t \geq 0$, the *time-t* map of a continuous dynamical system is the transformation of state space which takes $x(0)$ to $x(t)$.

Example 5. Let $k < 0$ in $\dot{x} = kx$, with $x(0) = x_0$. Here, the state space is $\mathbb{R}$ (the phase space, as opposed to the trajectory space $\mathbb{R}^2$), and the general solution is $\Phi(x_0, t) = x_0 e^{kt}$ (the evolution of the dynamical system is $\Phi(x, t) = xe^{kt}$. Notice that

$$\Phi(x, 0) = x, \quad \text{while} \quad \Phi(x, 1) = e^k x.$$

Hence the time-1 map is simply multiplication by $r = e^k$. The time-1 map is the discrete dynamical system on $\mathbb{R}$ given by the function above $f(x) = rx$. In this case, $r = e^k$, where $k < 0$, so that

$$0 < r = e^k < 1.$$

See Figure 1. Now how do the orbits behave?

Question 6. *Given any dynamical system, describe the time-0 map.*

Definition 7. For a discrete dynamical system $f : X \to X$, a *fixed point* is a point $x_* \in X$, where $f(x_*) = x_*$, or where

$$\mathcal{O}_x = \left\{x_*, x_*, x_*, \dots\right\}.$$

The orbit of a fixed point is also called a trivial orbit. All other orbits are called non-trivial.

In our example above, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^k x$, $k < 0$, we have $x = 0$ as the ONLY fixed point. This corresponds nicely with the unique particular solution to the ODE $\dot{x} = kx$ corresponding to the equilibrium $x(t) \equiv 0$.

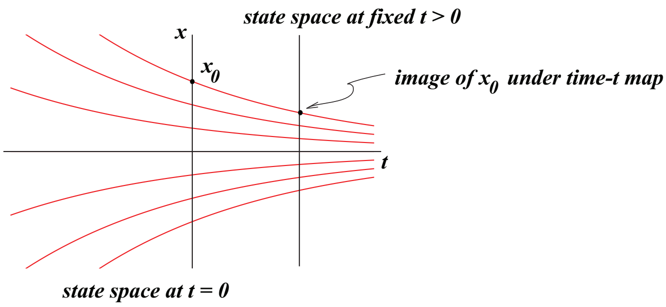

FIGURE 2. The time-t map for some positive time of $\dot{x} = kx$, $k < 0$.

So what else can we say about the "structure" of the orbits? That is, what else can we say about the "dynamics" of this dynamical system? For starters, the forward orbit of a given x_0 will look like the graph of the discrete function $f_{x_0} : \mathbb{N} \to \mathbb{R}^2$, $f_{x_0}(n) = x_0 e^{kn}$. Notice how this orbit follows the trajectory of x_0 of the continuous dynamical system $\dot{x} = kx$. Here, f is the time-1 map of the ODE. Notice also that, as a transformation of phase space (the x-axis), f is not just a continuous function but a differentiable one, with $0 < f'(x) = e^k < 1$, $\forall x \in \mathbb{R}$. The orbit of the fixed point at $x = 0$, as a sequence, certainly converges to 0. But here ALL orbits have this property, and we can say

$$\forall x \in \mathbb{R}, \quad \lim_{n\to\infty} \mathcal{O}_x = 0, \text{ or } \mathcal{O} \longrightarrow 0.$$

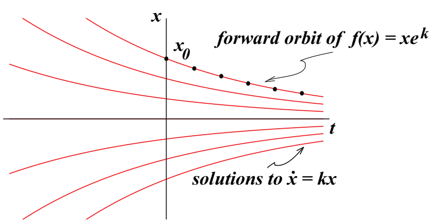

FIGURE 3. The forward orbit of $f(x) = xe^k$ lives on a solution to $\dot{x} = kx$, $k < 0$.

This gives a sense of what we will mean by a dynamical system exhibiting *simple dynamics*: If with very little effort or additional structure, one can completely describe the nature of all of the orbits of the system. Here, there is one fixed point, and all orbits converge to this fixed point.

Definition 8. For a discrete dynamical system, a smooth curve (or set of curves) ℓ in state space is called an *orbit line* if $\forall x \in \ell$, $\mathcal{O}_x \subset \ell$.

Example 9. The orbit lines for time-t maps of ODEs are the trajectories of the ODE.

Exercise 1. Go back to Figure 1. Describe completely the orbit structure of the discrete dynamical system

$f(x) = rx$ for other two cases, when $r = 1$ and $r > 1$ (corresponding to $r = e^k$, for $k = 0$ and $k < 0$, respectively). That is, classify all possible different types of orbits, in terms of whether they are fixed or not, where they go as sequences, and such. You will find that even here, the dynamics are simple, but at least for the $k > 0$ case, one has to be a little more careful about where orbits go.

Exercise 2. As in the previous exercise, now describe the dynamics of the discrete dynamical system $f(x) = rx$, when $r < 0$ (again, there are cases here). In particular, what are the orbit lines here? You will find that this case does NOT correspond to a time-t map of the ODE $\dot{x} = kx$ for any value of k (why not?) Can you construct an ODE that DOES have $f(x) = rx$ as its time-1 map, for a given value of $r < 0$?

Exercise 3. For the discrete dynamical system $f : \mathbb{R} \to \mathbb{R}$, $f(x) = rx + b$, completely describe the orbit structure when $b \neq 0$, and $0 < r < 1$.

Exercise 4. Given $\dot{x} = f(x)$, $f \in C^1(\mathbb{R})$, an equilibrium solution is defined as a constant function $x(t) \equiv c$ which solves the ODE, and they can be found by solving $f(x) = 0$ (remember this?) Instead, define an equilibrium solution $x(t)$ as follows: A solution $x(t)$ to $\dot{x} = f(x)$ is called an *equilibrium solution* if there exists $t_1 \neq t_2$ in the domain of $x(t)$ where $x(t_1) = x(t_2)$. Show that this new definition is equivalent to the old one.

These are all good questions to explore. For now, the above example $f(x) = e^k x$, where $k < 0$ is an excellent example of a particular class of dynamical systems:

Definition 10. a *metric* on a subset of Euclidean space $X \subset \mathbb{R}^n$ is a function $d : X \times X \to \mathbb{R}$ where

(1) $d(x, y) \geq 0$, $\forall x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$, $\forall x, y \in X$.
(3) $d(x, y) + d(y, z) \geq d(x, z)$, $\forall x, y, z \in X$.

One such choice of metric is the "standard Euclidean distance" metric

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Note that for $n = 1$, this metric reduces to $d(x, y) = \sqrt{(x - y)^2} = |x - y|$.

Remark 11. When discussing points in Euclidean space, it is conventional to denote scalars (elements of $\mathbb{R}$) with a variable in italics, and vectors (elements of $\mathbb{R}^n$, $n > 1$) as a variable in boldface. Thus $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. In the above definition of a metric, we didn't specify whether X was a subset of $\mathbb{R}$ or something larger. In the absence of more information regarding a space X, we will always use simple italics for its points, so that $x \in X$, even if it is possible that $X = \mathbb{R}^5$, for example. We will only resort to the vector notation when it is assured that we are specifically talking about vectors of a certain size. This is common in higher mathematics like topology.

Definition 12. A map $f : X \to X$, where $X \subset \mathbb{R}^n$ is called *Lipschitz continuous* (with constant λ), or λ-Lipschitz, if

$$d\left(f(x), f(y)\right) \leq \lambda d(x, y), \quad \forall x, y \in X. \tag{1}$$

Some notes:

- λ is a bound on the stretching ability (comparing the distances between the images of points in relation to the distance between their original positions) of f on X. This is actually a form of smoothness stronger than continuity.
- To get a sense for what Lipschitz continuity is saying, consider the following: On a bounded interval in $\mathbb{R}$, polynomials are always Lipschitz continuous. In fact, the total finite length of the range in this case, is a Lipschitz constant. Rational functions, on the other hand, even though they are continuous and differentiable on their domains, are not Lipschitz continuous on any interval whose closure contains an asymptote.
- It should be obvious that $\lambda > 0$. Why?
- We can define
$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{d\left(f(x), f(y)\right)}{d(x, y)},$$
which is the infimum of all λ's that satisfy Equation 1. When we speak of specific values of λ for a λ-Lipschitz function, we typically use $\lambda = \mathrm{Lip}(f)$, if known.

Definition 13. A λ-Lipschitz function $f : X \to X$ on a metric space X is called a *contraction* if $\lambda < 1$.

Proposition 14. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on the interval I, where $\forall x \in I$, we have $|f'(x)| \leq \lambda$. Then f is λ-Lipschitz.*

Proof. Really, this is simply an application of the Mean Value Theorem: For a function f differentiable on a bounded, open interval (a, b) and continuous on its closure, there is at least one point $c \in (a, b)$ where $f'(c) = \frac{f(b)-f(a)}{b-a}$, the average total change of the function over $[a, b]$. Here then, for any $x, y \in I$ (thus ALL of $[x, y] \in I$ even when I is neither closed nor bounded), there will be at least one $c \in I$ where

$$d\left(f(x), f(y)\right) = |f(x) - f(y)| = |f'(c)||x - y| < \lambda|x - y| = \lambda d(x, y).$$

□

Example 15. Back to the previous example $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^k x$, the time-1 map of the ODE $\dot{x} = kx$. Given that $f'(x) = e^k$ everywhere, in the case that $k < 0$, the map f is a contraction on ALL of $\mathbb{R}$. Can you see this?

Proposition 16. *Let $f : I \to I$ for I a closed, bounded interval, and f continuously differentiable with $|f'(x)| < 1$ $\forall x \in I$. Then f is a contraction.*

Proof. $f'(x)$ is continuous on I so it will achieve its maximum there by the Extreme Value Theorem, and

$$\max_{x \in I} |f'(x)| = \lambda < 1.$$

□

Note. *If I is not closed, or is not bounded, this may NOT be true. Think about why not.*

Exercise 5. Show $f(x) = 2\sqrt{x}$ is NOT a contraction on $(1, \infty)$.

Definition 17. For $f : X \to X$ a map, a point $x \in X$ is called *periodic* (with period n) if $\exists n \in \mathbb{N}$ such that $f^n(x) = x$. The smallest such natural number is called the *prime period* of x.

Notes:

- If $n = 1$, then x is a fixed point.
- Define

$$\mathrm{Fix}(f) = \left\{ x \in X \;\middle|\; f(x) = x \right\}$$
$$\mathrm{Per}_n(f) = \left\{ x \in X \;\middle|\; f^n(x) = x \right\}$$
$$\mathrm{Per}(f) = \left\{ x \in X \;\middle|\; \exists n \in N \text{ such that } f^n(x) = x \right\}.$$

Keep in mind that these sets are definitely not mutually exclusive.