110.421 DYNAMICAL SYSTEMS

Week 1 Lecture 1 Notes

1. What is a dynamical system?

Probably the best way to begin this discussion is with arguably a most general and yet least helpful statement:

Definition 1. A *dynamical system* is a mathematical formalization for any fixed rule which describes the dependence of a point's position in some ambient space on a parameter.

- The parameter here is usually called "time", and can be
  (1) discrete (think the integers $\mathbb{Z}$), or
  (2) continuous (defined by some interval in $\mathbb{R}$).
  It can also be much more general, taking values as subsets of $\mathbb{C}$, $\mathbb{R}^n$, the quaternions, or indeed any set with the structure of a group. However, classically speaking, a dynamical system really involves a parameter that takes values only in a subset of $\mathbb{R}$. We will hold to this convention.
- The ambient space has a state to it which changes as one varies the parameter. Roughly, every point has a position relative to the other points; coordinates often provide this position idea, but in general this is a notion of a topology on a set: A topology gives a set the mathematical property of a space.) We call this ambient space the *state space*: it is the set of all possible states a dynamical system can be in at any moment.
- The fixed rule is usually a way for going from one state to the next, given as a recursively defined function. In contrast, an *evolution* is a way of going from any particular state to any other state reachable from that initial state. AS we will see, such a function will in general NOT be known *a priori*.

While this idea of a dynamical system is general enough to be useless, it will be instructive. Before tightening it up, let's look at some classical examples:

1.1. Ordinary Differential Equations.

Given the vector-ODE in $\mathbb{R}^n$,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

a solution, if it exists, is a vector of functions $\mathbf{x}(t) = [x_1(t)\ \ x_2(t)\ \ \cdots\ \ x_n(t)]^T$ parameterized by a real variable $t \in \mathbb{R}$ where t is valid on some interval. Here:

- The ODE itself is the fixed rule, describing the infinitesimal way to go from one state to the next by a slight continuous variation in the value of the parameter t. Solving the ODE means finding the unknown function $\mathbf{x}(t)$. The initial data provide an initial state of the variables of the system. Then $\mathbf{x}(t)$, for valid values of t, provides the various "other" states of the system as compared to the initial state. Collecting up

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all the functions $\mathbf{x}(t)$ for all valid sets of initial data (basically, finding the expression that writes the constants of integration of the general solution to the ODE in terms of the initial data), into one big function IS the evolution.

- This type of a dynamical system is called *continuous*, since the parameter t will take values in some domain (an interval) in $\mathbb{R}$. Dynamical systems like this arising from ODEs are also called *flows*, since the various IVP solutions in phase space look like the flow lines of a fluid in phase space flowing along the slope field (vector field defined by the ODE).
- In this particular example, the state space is (a subset of) $\mathbb{R}^n$, where the solutions live as parameterized curves, are called trajectories. We also call this space the *phase space*.

So ODEs are examples of continuous dynamical systems (actually differentiable dynamical systems). Solving the ODE (finding the vector of functions $\mathbf{x}(t)$, means finding the rule which stipulates any future state of a point given a starting state. But as we will see, when thinking of ODEs as dynamical systems, we have a different perspective on what we are looking for in solving the ODE.

1.2. **Maps.** Given any set X and a function $f : X \to X$ from X to itself, one can form a dynamical system by simply applying the function over and over (iteratively) to X. When the set has a topology on it (a mathematically precise notion of an "open subset", which we will get to), we can then discuss whether the function f is continuous or not. When X has a topology, it is called a space, and a continuous function $f : X \to X$ is called a *map*.

- We will always talk of sets as spaces, detailing the topology only as needed. Here the *state space* is X, with the positions of its points given by the topology (in almost all situations, this will be obvious in context).
- the fixed rule is the map f, which is also sometimes also called a cascade.
- Here f defines the evolution (but only recursively) by composing f with itself: Given $x \in X$, define $x_0 = x$, and $x_1 = f(x_0)$. Then
$$x_2 = f(x_1) = f(f(x_0) = f^2(x_0),$$
and for all $n \in \mathbb{N}$, (the natural numbers)
$$x_n = f(x_{n-1}) = f(f(x_{n-2})) = \overbrace{f(f(\cdots f(f(x_0))\cdots))}^{n \text{ times}} = f^n(x_0).$$
- Maps are examples of *discrete dynamical systems*. Some examples of discrete dynamical systems you may have heard of include discretized ODEs, including difference equations and time-t maps, and fractal constructions like Julia sets and the associated Mandelbrot arising from maps of the complex plane to itself. Some objects that are not considered to be constructed by dynamical systems (at least not directly)

include fractals like Sierpinski's carpet, Cantor sets, and Fibonacci's Rabbits (given by a second order recursion). Again, we will get to these.

Besides these classic ideas of a dynamical system, there are much more abstract notions of a dynamical system:

1.3. **Symbolic Dynamics.** Given a set of symbols $M = \{A, B, C, \dots\}$, consider the "space" of all bi-infinite sequences of these symbols (infinite on both sides)

$$\Omega_M = \left\{ (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) \middle| \, i \in \mathbb{Z}, x_i \in M \right\}.$$

This is called the sequence space for M. Now let $f : \Omega_M \to \Omega_M$ be the *shift* map: on each sequence, it simply takes $i \mapsto i + 1$; each sequence goes to another sequence which looks like a shift of the original one.

Note. *We can always consider this (very large) set of infinite sequences as a space once we give it a topology like I mentioned. This would involve defining open subsets for this set, and we can do this through ϵ-balls be defining a notion of distance between sequences (a metric). For those who know analysis, what would be a good metric for this set to make it a space using the metric topology? For now, simply think of this example as something to think about. Later and in context, we will focus on this type of dynamical system and it will make more sense.*

This discrete dynamical system is sometimes used as a new dynamical system to study the properties of an old dynamical system whose properties were hard to study. We will revisit this later.

Sometimes, in a time-dependent system, the actual dynamical system will need to be constructed before it can be studied.

1.4. **Billiards.** Consider two point-beads moving at constant (possibly different) speeds along a finite length wire, with perfectly elastic collisions both with each other and with the walls. A state of this system will be the positions of each of the beads at a given moment of time.

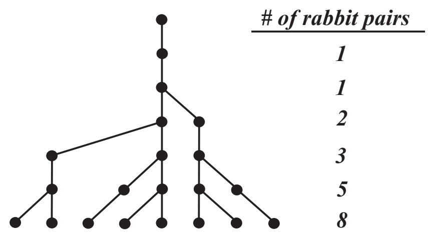
Exercise 1. One way to view the state space is as a triangle in the plane. Work this out. What are the vertices of this triangle? Does it accurately describe ALL of the states of the system? And once you correctly describe the state space, what will motion look like in it? How does the dynamical system evolve?

Now consider a point-ball moving at a constant velocity inside a closed, bounded region of $\mathbb{R}^2$, where the boundary is smooth and collisions with the boundary are perfectly elastic. Questions:

(1) How does the shape of the region affect the types of paths the ball can traverse?
(2) Are there closed paths (periodic ones)?
(3) can there be a dense path (one that eventually gets arbitrarily close to any particular point in the region?

There is a method to study this type of dynamical system by creating a discrete dynamical system to record movement. In this discrete dynamical system, regardless of the shape of the region, the state space is a cylinder. Can you see it? If so, what would be the evolution?

1.5. **Other recursions.** The Rabbits of Leonardo of Pisa: A beautiful example of a type of growth that is not exponential, but something called asymptotically exponential. We will explore this more later. For now, place a newborn pair of breeding rabbits in a closed environment. Rabbits of this species produce another pair of rabbits each month after they become fertile (and they never die nor do they experience menopause). Each new pair of rabbits (again, neglect the incest, gender and DNA issues) becomes fertile after a month and starts producing each month starting in the second month. How many rabbits are there after 10 years?



Month	a_n	j_n	b_n	total pairs
1	0	0	1	1
2	0	1	0	1
3	1	0	1	2
4	1	1	1	3
5	2	1	2	5
6	3	2	3	8
7	5	3	5	13

Given the chart in months, we see a way to fashion an expression governing the number of pairs at the end of any given month: Start with r_n, the number of pairs of rabbits in the nth month. Rabbits here will come in three types: Adults a_n, juveniles j_n, and newborns b_n, so that $r_n = a_n + j_n + b_n$. Looking at the chart, we can see that there are constraints on these numbers:

(1) the number of newborns at the $(n+1)$st stage equals the number of adults at the nth stage plus the number of juveniles at the nth stage, so that

$$b_{n+1} = a_n + j_n.$$

(2) This is also precisely equal to the number of adults at the $(n+1)$st stage, so that

$$a_{n+1} = a_n + j_n.$$

(3) and finally, the number of juveniles a the $(n+1)$st stage is just the number of newborns at the nth stage, so that

$$j_{n+1} = b_n.$$

Thus, we have

$$r_n = a_n + j_n + b_n = (a_{n-1} + j_{n-1}) + b_{n-1} + (a_{n-1} + j_{n-1}).$$

And since in the last set of parentheses, we have $a_{n-1} = a_{n-2} + j_{n-2}$ and $j_{n-1} = b_{n-2}$, we can substitute these in to get

$$\begin{aligned} r_n = a_n + j_n + b_n &= (a_{n-1} + j_{n-1}) + b_{n-1} + (a_{n-1} + j_{n-1}) \\ &= a_{n-1} + j_{n-1} + b_{n-1} + a_{n-2} + j_{n-2} + b_{n-2} = r_{n-1} + r_{n-2}. \end{aligned}$$

Hence the pattern is ruled by a second-order recursion $r_n = r_{n-1} + r_{n-2}$ with initial data $r_0 = r_1 = 1$. Being a second order recursion, we cannot go to the next state from a current state without also knowing the previous state. This is an example of a model which is not a dynamical system. We can make it one, but we will need a bit more structure, which we will introduce later.

Here is a much better definition of a dynamical system:

Definition 2. A dynamical system is a triple $(\mathcal{S}, \mathcal{T}, \Phi)$, where $\mathcal{S}$ is the state space (or phase space), $\mathcal{T}$ is the parameter space, and

$$\Phi : (\mathcal{S} \times \mathcal{T}) \longrightarrow \mathcal{S}$$

is the evolution.

Some notes:

- In the previous discussion, the fixed rule was a map or an ODE which would only define recursively what the evolution would be. In this definition, Φ defines the entire system, mapping where each point $s \in \mathcal{S}$ goes for each parameter value $\tau \in \mathcal{T}$. It is the functional form of the evolution, unraveling the recursion and allowing you to go from a starting point to any point reachable by that point given a value of the parameter.
- In ODEs, Φ plays the role of "solving" for the *general* solution, as a 1-parameter family of solutions (literally a 1-parameter family of transformations of phase space): In this general solution, one knows for ANY specified starting value where it will be for ANY valid parameter value, all in one function of two variables.

  Example 3. In the Malthusian growth model, $\dot{x} = kx$, with $k \in \mathbb{R}$, and $x(t) \geq 0$ a population, the general solution is given by $x(t) = x_0 e^{kt}$, for $x_0 \in \mathbb{R}_+$ (the nonnegative real numbers: Really, the model works for $x_0 \in \mathbb{R}$, but if the model represents population growth, then initial populations can ONLY be nonnegative, right?) the unspecified initial value at $t = 0$. Here, $\mathcal{S} = \mathbb{R}_+$, $\mathcal{T} = \mathbb{R}$ and $\Phi(s,t) = se^{kt}$.

  Example 4. Let $\dot{x} = -x^2 t$, $x(0) = x_0 > 0$. Using the technique known as separation of variables, we can integrate to find an expression for the general solution as $x(t) = \frac{1}{\frac{t^2}{2} + C}$. And since $x_0 = \frac{1}{C}$ (you should definitely DO these calculations explicitly!), we get

$$\Phi(x_0, t) = \frac{1}{\frac{t^2}{2} + \frac{1}{C}} = \frac{2x_0}{x_0 t^2 + 2}.$$

Here again, we choose $cS = \mathbb{R}_+$, and $\mathcal{T} = \mathbb{R}$. Question: Do you see any issues with allowing $x_0 < 0$? Let $x_0 = -2$, and describe the particular solution on the interval $t \in (0, 2)$.

- In discrete dynamics, for a map $f : X \to X$, we would need a single expression to write $\Phi(x, n) = f^n(x)$. This is not always easy or doable, as it would involve finding a functional form for a recursive relation. Try doing this with f a a general polynomial of degree more than 1.

  Example 5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = rx$, for $r \in \mathbb{R}_+$. Then $\Phi(x, n) = r^n x$.

  Example 6. For Leonardo of Pisa's (also known as Fibonacci, by the way) rabbits, we will have to use the recursion to calculate every month's population to get to the 10-year mark. However, if we could find a *functional* form for the recursion, giving population in terms of month, we could than simply plug in $12 \cdot 10 = 120$ months to calculate the population after 10 years. The latter functional form is the evolution Φ in the definition of a dynamical system above. How does one find this? We will see.

Exercise 2. Find a closed form expression for the evolution of $f(x) = rx + a$, in the case where $-1 < r < 1$ and a are constants. Also determine the unique point where $f(x) = x$ in this case.

In general, finding Φ (in essence, solving the dynamical system) is very hard if not impossible, and certainly often impractical. Many times the purpose of studying a dynamical system is not to actually solve it. Rather, it is to gain insight as to the structure of its solutions. Really, we are trying to make *qualitative* statements about the system rather than quantitative ones. Think about what you did when studying nonlinear systems of first order ODEs in 110.302. Think about what you did when studying autonomous first order ODEs.

Here is another less rigorous definition of a dynamical system:

Definition 7. Dynamical Systems as a field of study attempts to understand the structure of a changing mathematical system by identifying and analyzing the things that do not change.

Some ideas:

- Invariance: First integrals, and phase space area under a conservative vector field and its flow.
- Symmetry: Periodicity and possible reduction of order techniques.
- Asymptotics: Long-term behavior of solutions, equilibria and limit cycle stability.

Example 8. In an exact differential equation

$$M(x,y)\,dx + N(x,y)\,dy = M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

we have $M_y = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = N_x$. We know then that there exists a function $\phi(x,y)$, where $\frac{\partial \phi}{\partial x} = M$ and $\frac{\partial \phi}{\partial y} = N$. Indeed, given a twice differentiable function $\phi(x,y)$ defined on a domain in the plane, it's level sets are equations $\phi(x,y) = C$, for C a real constant. Each level set defines y implicitly as a function of x. Thinking of y as tied to x implicitly, differentiate $\phi(x,y) = C$ with respect to x and get

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0.$$

This last equation will match the original ODE precisely if the two above properties hold. The interpretation then is: The solutions to the ODE correspond to the level sets of the function ϕ. We can say that that solutions to the ODE "are forced to live" on the level sets of ϕ. Thus, we can write the general solution set (at least implicitly) as $\phi(x,y) = C$, again a 1-parameter family of solutions. Here ϕ is a first integral of the flow given by the ODE.

Exercise 3. Solve the differential equation $12 - 3x^2 + (4-2y)\frac{dy}{dx} = 0$ and express the general solution in terms of the initial condition $y(x_0) = y_0$. This is your function $\Phi(x,y)$.

Example 9. Newton-Raphson: Finding a root of a (twice differentiable) function $f : \mathbb{R} \to \mathbb{R}$ leads to a discrete dynamical system $x_n = g(x_{n-1})$, where

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

One here does not need to actually solve the dynamical system (find a form for the function Φ). Instead, all that is needed are to satisfy some basic properties of f to know that if you start sufficiently close to a root, the long-term behavior of any starting point IS a root.

Example 10. Autonomous ODEs: One can definitely integrate the autonomous first-order ODE

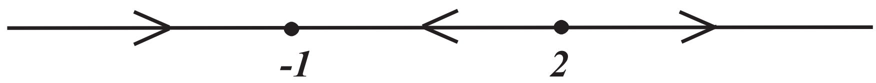
$$y' = f(y) = (y-2)(y+1), \quad y(0) = y_0,$$

since it is separable, and the integration will involve a bit of partial fraction decomposing. The solution is

$$y(t) = \frac{Ce^{3t} + 2}{1 - Ce^{3t}}. \tag{1}$$

Exercise 4. Calculate Equation 1 for the ODE in Example 10.

Exercise 5. Now find the evolution for the ODE in Example 10 (this means write the general solution in terms of y_0 instead of the constant of integration C.)



But really, is the explicit solution necessary? Once can simply draw the phase line, seeing that the equilibrium solutions occur at $y(t) \equiv -1$ and $y(t) \equiv 2$, and that the equilibrium at -1 is asymptotically stable (the one at 2 is unstable). Thus, if long-term behavior is all that is necessary to know, we have:

$$\lim_{t\to\infty} y(t) = \begin{cases} -1 & \text{if} \quad y_0 < 2 \\ 2 & \text{if} \quad y_0 = 2 \\ \infty & \text{if} \quad y_0 > 2. \end{cases}$$

In both these last two examples, actually solving the dynamical system isn't necessary to gain important information about the system.