MATH 421 DYNAMICS

Week 12 Lectures Notes

For $f : X \to X$, a continuous map on a metric space (X, d), consider a sequence of new metrics on X indexed by $n \in \mathbb{N}$:

$$d_n^f(x,y) := \max_{0 \le i \le n-1} d\left(f^i(x), f^i(y)\right).$$

Here, the new metrics d_n^f actually measure a "distance" between orbit segments

$$\mathcal{O}_{x,n} = \left\{x, f(x), \ldots, f^{n-1}(x)\right)$$
$$\mathcal{O}_{y,n} = \left\{y, f(y), \ldots, f^{n-1}(y)\right)$$

as the farthest that these two sets diverge along the orbit segment, and assigns this distance to the pair x and y.

Exercise 1. Show for a given n that d_n^f actually defines a metric on X.

Now, using the metric d_n^f, we can define an r-ball as the set of all neighbor points y whose nth orbit-segment $\mathcal{O}_{y,n}$ stays within r distance of $\mathcal{O}_{x,n}$:

$$B_r(x,n,f) = \left\{ y \in X \,\middle|\, d_n^f(x,y) < r \right\}.$$

Convince yourself that as we increase n, the orbit segment is getting longer, and more and more neighbors y will have orbit segments that move away from $\mathcal{O}_{x,n}$. Thus the r-ball will get smaller as n increases. But by continuity, the r-balls for any n will always be open sets in X that have x as an interior point. Also, as r goes to 0, the r-balls will also get smaller, right?

Now define the r-capacity of X, using the metric d_n^f and the new r-balls $B_r(x,n,f)$, denoted $S_{(X,d)}(r,n,f)$ (this is the SAME notion of r-capacity as the one we used for the box dimension! We are only changing the metric on X to d_n^f. But the actual calculations of the r-capacity depend on the choice of metric). As before, as r goes to 0, the r-balls shrink, and hence the r-capacity grows. And also, as n goes to ∞, we use the different d_n^f to measure ultimately the distances between entire positive orbits. This also forces the r-balls to shrink, and hence the r-capacity to grow. What is the exponential growth rate of the r-capacity as $r \to 0$? This is the notion of topological entropy:

Definition 1. Let $h_d(f,r) := \overline{\lim}_{n\to\infty} \dfrac{\log S_d(r,n,f)}{n}$. Then

$$h_d(f) := \lim_{r\to 0} h_d(f,r)$$

is called the *topological entropy* of the map f on X.

There are many things to say about this. To start:

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- Topological entropy is a measure of the tendency of orbits to diverge from each other. It will always be a non-negative number, and the higher it is, the faster orbits are diverging. In Euclidean space, maybe this is not so special (think of the linear map on $\mathbb{R}^2$ given by the matrix λI_2, with $\lambda > 1$. All orbits diverge, but the dynamics is not very interesting), but in a compact space with all orbits diverging, the resulting messy nature of the dynamics can be quite interesting. Thus, topological entropy is a measure of the orbit complexity, and the higher the number, the more interesting (read messy) the dynamical structure.
- Another common notation for topological entropy is $h_{top}(f)$ or $h_T(f)$ or even $h(f)$. These are, in a sense, more accurate since it turns out that the topological entropy of a map does not actually depend on the metric d, at least up to equivalence, chosen for use in its definition. It is possible, however, that inequivalent metrics may lead to either the same or a different entropy. We will use the notation $h(f)$ in our subsequent discussion.
- Contractions and isometries have no entropy:

  Proposition 2. *Let f be either a contraction or an isometry. Then $h(f) = 0$.*

  Proof. In the case of f an isometry, for any $n \in \mathbb{N}$, $d_n^f = d$, since distances between iterates of a map are the same as the original distances between the initial points. Hence the r-capacity $S_{(X,d)}(r,n,f) = S_{(X,d)}(r,f)$ does not depend on n, and hence $h(r,f) = 0$. For a contraction, the iterates of two distinct points are always closer together than the original points. Hence also here $d_n^f = d$. This leads to the same conclusion. □

- Topological entropy is a dynamical invariant (invariant under conjugacy). This means that if f is (semi-)conjugate to g, then $h(f) = h(g)$. However, it is also useful to use the contrapositive: If one has two maps where $h(f) \neq h(g)$, then it is not possible that f is (semi-)conjugate to g.
- topological entropy measures, in a way, the exponential growth rate of the number of trajectories that are r-separable after n iterations. Suppose this number is proportional to e^{nh}. Then h would be the growth rate for a fixed r, and as $r \to 0$, this h would tend to the entropy.
- defining the topological entropy for a flow is simply a matter of replacing the $n \in \mathbb{N}$ with $t \in \mathbb{R}$ in all of the definitions for the invariant. we can relate the two in a way: The topological entropy of a flow is equal to the topological entropy of its time-1 map (really, its time-t for any choice of t, since the flow provides the conjugacy of any t-map with any other).
- In practice, topological entropy is quite hard to calculate. However, in many cases, and in response to the last bullet point, the entropy is directly related to the largest Lyapunov exponent of the system, at least for C^1 systems.

Proposition 3. *For the expanding map $E_m : S^1 \to S^1$, where $E_m(x) = mx \mod 1$, and $|m| \geq 1$, $h(E_m) = \log |m|$.*

Proposition 4. *For* $f : \mathbb{T}^2 \to \mathbb{T}^2$, *given by* $\vec{x} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$ *(this was the map* F_L *from before),* $h(f) = \log \frac{3+\sqrt{5}}{2}$.

Note: In both of these cases, the topological entropy of the map IS the maximum positive Lyapunov exponent of the system.

Example 5. Show that $h(E_2) = \log 2$.

To do this calculation, we will need to quantify the r-capacity of S^1 under this map. This amounts to calculating $S_{(S^1,d)}(r, n, E_2)$ for a fixed r and as a function of the iterate number n. Hence we start with a good idea of what constitutes the actual size of an r-ball $B_r(x, n, E_2)$ for a choice of n. Note first that by its definition, $B_r(x, n, E_2)$ is the set of points whose distance away from x is less than r after n iterates of E_2. As the map is expanding by a factor of 2 (locally), distances double after each iterate (see the figure). Hence we will have to get closer to x when we start iterating to remain within r as we iterate. Hence $B_r(x, n, E_2)$ will shrink in size as n increases. How will it shrink?

Suppose for a minute that $r = \frac{1}{4}$. Choose an $x \in S^1$, and recall that

$$B_{\frac{1}{4}}(x, 0, E_2) = \left\{ y \in S^1 \,\middle|\, d_0^{E_2}(x, y) = d(x, y) = |x - y| < \frac{1}{4} \right\}.$$

The radius of $B_r(x, n, E_2)$ is $\frac{1}{4}$ here. After one iterate, however,

$$B_{\frac{1}{4}}(x, 1, E_2) = \left\{ y \in S^1 \,\middle|\, d_1^{E_2}(x, y) = \max\{|x - y|, |2x - 2y|\} < \frac{1}{4} \right\}.$$

Here, it is obvious that the condition that $d_1^{E_2}(x, y) = |2x - 2y| = 2|x - y| < \frac{1}{4}$ means that the actual distance between x and y would have to be $|x - y| < \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$. Hence the radius of $B_{\frac{1}{4}}(x, 1, E_2)$ is only $\frac{1}{8}$. Similarly, the radius of $B_{\frac{1}{4}}(x, 2, E_2)$ is only $\frac{1}{16}$, and in general we have that

$$\text{radius}\left(B_{\frac{1}{4}}(x, n, E_2)\right) = \frac{1}{4} \cdot \frac{1}{2^n}.$$

But, really, the initial size of r does not determine the relative sizes of the r-balls with respect to each other. Hence, we can say that, for any choice of $r > 0$, we have

$$\text{radius}\left(B_r(x, n, E_2)\right) = r \cdot \frac{1}{2^n}.$$

Recall that the r-capacity, $S_{(S^1,d)}(r, n, E_2)$ is the minimum number of the r-balls $B_r(x, n, E_2)$ it takes to cover S^1. Think of S^1 as being parameterized by the unit interval $[0, 1]$ with the identification of 0 and 1. Then we really only need to find out how many r-balls we need for a given iterate n to cover an interval of length 1. Call this number K_n. Hence, we solve the equation (really, it is an inequality, but since adding one more ball to each quantity will not change the limit, this is an okay simplification)

$$\#\Big(B_r\left(x,n,E_2\right)\Big)\cdot 2\cdot \text{radius}\Big(B_r\left(x,n,E_2\right)\Big) = K_n\cdot 2\cdot r\cdot \frac{1}{2^n} = 1.$$

Which is solved by $K_n = \frac{1}{r}\cdot 2^{n-1}$. This is $S_{(S^1,d)}\left(r,n,E_2\right)$.

We now calculate

$$\begin{aligned} h\left(E_2,r\right) &= \overline{\lim_{n\to\infty}} \frac{\log S_{(S^1,d)}\left(r,n,E_2\right)}{n} \\ &= \lim_{n\to\infty} \frac{\log \frac{1}{r}\cdot 2^{n-1}}{n} \\ &= \lim_{n\to\infty} \left(\frac{\log \frac{1}{r}}{n} + \frac{\log 2^{n-1}}{n}\right) \\ &= 0 + \log 2\cdot\left(\lim_{n\to\infty}\frac{n-1}{n}\right) = \log 2. \end{aligned}$$

Here again, the r-topological entropy does not depend on r at all, so that

$$h\left(E_2\right) = \lim_{r\to 0} h\left(E_2,r\right) = \lim_{r\to 0}\log 2 = \log 2.$$

1. Quadratic Maps (revisited)

We begin today by going back to quadratic maps:

Let $I = [0,1]$ and $f_\lambda : I \to I$, $f_\lambda(x) = \lambda x(1-x)$, but this time let $\lambda \in [3,4]$.

Definition 6. Let $x \in X$ be fixed for the map $f : X \to X$. The *basin of attraction* of x is

$$B(x) = \left\{ y \in X \,\middle|\, \mathcal{O}_y \to x \right\}.$$

- Sometimes the basin of attraction is easy to describe:

  Example 7. Let $\dot{r} = r(r-1)$, $\dot{\theta} = 1$ be the planar ODE system. It should be obvious now that the only equilibrium solution is at the origin of the plane, and the only other "interesting" behavior is the unstable limit cycle given by the equation $r(t) \equiv 1$. Since solutions are unique on all of $\mathbb{R}^2$ (and hence cannot cross), what starts inside the unit circle stays inside. And since the limit cycle is repelling, and there are no other limit cycles or equilibria inside the unit circle, it must be the case that the origin is attracting (you can also see this directly by noting that $\dot{r} < 0$, $\forall r \in (0,1)$). hence the basin of attraction of the origin is the open unit disk

  $$B\left((0,0)\right) = \left\{ (r,\theta \in \mathbb{R}^2 \,\middle|\, r < 1 \right\}.$$

- Sometimes, it is not:

  Example 8. Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2 + c$, for $c \in \mathbb{C}$ a constant. For $c = 0$, we get a rather plain model. $\mathcal{O}_z \to 0$ $\forall |z| < 1$, and $\mathcal{O}_z \to \infty$ $\forall |(|z) > 1$. Do you recognize the map on the unit circle $|z| = 1$? It is the expanding (and chaotic) map $E_2 : S^1 \to S^1$ from before.

  Definition 9. For $P : \mathbb{C} \to \mathbb{C}$ a polynomial map, the *Julia Set* is the closure of the set of repelling periodic points of P.

  Keep this in mind. For the map E_2 in the circle, recall that the periodic points are dense in S^1 (this was a feature of chaos). And since the map is expanding, you can show that all of these periodic points are actually repelling (simultaneously!). The resulting mess is actually what a "sensitive dependence on initial conditions" is all about. Here again, the origin in $\mathbb{C}$ is an attracting fixed point, and its basin of attraction is everything inside the unit circle.

  Now, though, let c be small and non-zero. There will still be two fixed points, right? (think of solving the equation $z = f(z) = z^2 + c$. The solutions will be $z = \frac{1 \pm \sqrt{1-4c}}{2}$. For $z \in \mathbb{C}$, this always has two solutions!) The one near the origin will still be attracting, while the one near the unit circle will still be a part of a set of repelling periodic points whose closure will form a (typically) fractal structure. This is again the Julia Set for this value of c, and can be highly bizarre looking. I showed you a few examples in class.

  In sum, for general $c \in \mathbb{C}$, the Julia set is not a smooth curve. For example, let $c < -2$ be real. Then $f_c(z) = z^2 + c$ is topologically conjugate to a map of the form $x \mapsto \lambda x(1-x)$ for $\lambda > 4$ (this conjugacy is really just a change of variables. Can you find it?) The ramifications of this being that 1) the dynamics outside of the Julia Set are rather simple (think that outside of those interesting orbits of f_λ that stay within I forever, all orbits basically go to $-\infty$). But this implies that that Julia Set is conjugate to a Cantor Set. But this also means that the Cantor Set of points whose orbits stay within I under f_λ, $\lambda > 4$, consists of the closure of a set of repelling periodic points.

Definition 10. An m-periodic point p is called *attracting* under a continuous map f if $\exists \epsilon > 0$ such that $\forall x \in X$, where $d(x.p) < \epsilon$, then $d\left(f^n(x), f^n(p)\right) \overset{n\to\infty}{\longrightarrow} 0$.

Exercise 2. Show that for an attracting m-periodic point p, each distinct point it its orbit is also attracting.

Call the basin of attraction for an m-periodic point p the union of the basins of attraction for each point of $\mathcal{O}_p$. That is, for $\mathcal{O}_p = \{p.f(p), f^2(p), \dots, f^{m-1}(p)\}$, the basin of attraction of p is

$$B(p) = \left\{ x \in X \,\middle|\, d(f^n(x), f^{n+k}(p)) \overset{n\to\infty}{\longrightarrow} 0 \text{ for some } k \in \mathbb{N} \right\}.$$

Proposition 11. *Let $f : [a,b] \to [a,b]$ be C^2 and concave, where $f(a) = f(b) = a$. Then f has at most one attracting periodic orbit.*

We will not prove this. However, it is useful.

Example 12. As in the figure, in all of our examples of $f_\lambda : I \to I$, where $\lambda \in [0,3]$ there was always an attracting fixed point. However, for $\lambda = 3.1$, for example, the fixed point at $x = 0$ is repelling, and there is an attracting period-2 orbit (can you find the numeric values for this orbit?)

Theorem 13. *If f_λ has an attracting periodic orbit, then the set outside of the basin of attraction (called the universal repeller) is a nowhere-dense null set.*

Some notes:

- A nowhere dense null set in a metric space is a set that can be covered by balls whose total volume is less than ϵ.
- What can lie within the universal repeller? First, any repelling fixed or periodic points, of course. But since the logistic map is a two-to-one map, the pre-image of a fixed point consists of two points, and includes a point that was not previously fixed.

  Example 14. Let $\lambda = 3.1 = \frac{31}{10}$. It can be shown that $f_{3.1}$ has an attracting period-2 orbit. And $x_\lambda = 1 - \frac{1}{\lambda} = 1 - \frac{1}{\frac{31}{10}} = \frac{21}{31}$ is fixed under $f_{3.1}$ and repelling (check this!) But the point $1 - x_\lambda = \frac{10}{31}$ also maps to $\frac{21}{31}$. In fact, the point $1 - x_\lambda$ is always the pre-image of the fixed point x_λ due to where it sits on the graph of f_λ. Both of these points are NOT in the basic of attraction of any periodic orbit. But also, $1 - x_\lambda$ is NOT a periodic point. It is an eventually fixed point, but that is different. Now the point $1 - x_\lambda$ also has two pre-images (find them), and these two pre-images also have two pre-images. In fact, there are a countable number of pre-images that eventually get mapped onto x_λ. All of this set lies outside of the basic of attraction of any attracting periodic orbit, when x_λ is repelling. This gives you an idea of what is considered part of the universal repeller. Now think about how this set of pre-images of x_λ sit inside the interval! If you think about it correctly, you start to see just how fractals are born.

Example 15. For $\lambda \in \left[3, 1+\sqrt{6}\right]$, there exists an attracting, period-2 orbit. The basin of attraction is everything except for the points 0, and $x_\lambda = 1 - \frac{1}{\lambda}$ and ALL of their pre-images.

Let's work out the situation: For $\lambda \in (2,3)$, 0 is a repelling fixed point, x_λ is an attracting fixed point, and there are no other periodic points. In contrast, for $\lambda \in \left(3, 1+\sqrt{6}\right)$, both $x = 0$, and x_λ are repelling fixed point, and there now exist an attracting period-2 orbit. This means that we have reached a bifurcation value for λ at $\lambda = 3$. This type of bifurcation is called a *period-doubling bifurcation*, and is visually a "pitchfork" bifurcation for the map

f_λ^2. See the figure. Analytically, what happens is that the value of $|f'(x_\lambda)| < 1$ for $\lambda < 3$ and $|f'(x_\lambda)| > 1$ for $\lambda > 3$. But these derivatives are negative, right? for the map f_λ^2, this means that the same thing happens but the derivative are positive!. Geometrically, this determines how the graph of f_λ^2 crosses the line $y = x$, and the crossing changes from over/under to under/over as we pass through the value $\lambda = 3$. And when the graph of f_λ^2 passes to the under/over configuration, it creates two new fixed points (for the f_λ^2 map). You do not see these new crossings in the original map f_λ because they are only period two points. You can cobweb them to see that they are there, though. The under/over crossing means that the derivative is greater than 1, and hence the map is expending near the fixed point (repelling). In contrast, the two new fixed points are over/under crossings, with a derivative less than 1, and hence are attracting. Again, see the figure. it is all in there.

Finally, what happens when $\lambda = 1 + \sqrt{6}$. Basically, the same thing, except that the period-2 orbit becomes a repelling orbit and a period-4 orbit is born and is attracting! Another period-doubling bifurcation.

Theorem 16. *There exists a monotonic sequence of parameter values*

$$\lambda_1 = 3, \quad \lambda_2 = 1 + \sqrt{6}, \quad \lambda_3 = \dots, \quad \textit{such that } \forall \lambda \in (\lambda_n, \lambda_{n+1}),$$

the quadratic map $f_\lambda(x) = \lambda x(1-x)$ has an attracting period-2^n orbit, two repelling fixed points at $x = 0, x_\lambda$, and one repelling period-2^k orbit for each $k = 1, \dots, n-1$.

Notes:

- This is called a *period-doubling cascade.*
- At every new λ_n, the previous attracting periodic orbit becomes repelling, and adds (with all of its pre-images) to the universal repeller.
- The length of the intervals $(\lambda_n, \lambda_{n+1})$ decrease exponentially as n increases, and go to 0 somewhere before $\lambda = 4$.
- In fact, one can calculate the exponential decay of these interval lengths:

  $$\delta = \lim_{n\to\infty} \frac{\text{length}\,(\lambda_{n-1}, \lambda_n)}{\text{length}\,(\lambda_n, \lambda_{n+1})} = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \cong 4.6992016010\cdots.$$

  This number has a universal quality to it, as it is always the exponential decay rate of the lengths between bifurcation values in period-doubling cascades. It is called the *Feigenbaum Number.*
- The full bifurcation diagram looks like the figure. At the bak edge of the cascade is a place called the transition to chaos. At this point, there are a countable number of repelling periodic points. This collection along with all of their pre-images wind up being dense in the interval, and hence cause a sensitive dependence on initial conditions, commonly found in chaotic systems. This is the Julia set, which in a chaotic system is the entire set.

- Note the self-similar structure of the bifurcation diagram. it is not a fractal, really, but it is related to many of them in interesting ways.
- Look carefully at the bifurcation diagram. Even after the transition to chaos, there seem to be regions of calm periodic behavior. These are not artifacts. In fact, there exists an attracting period-3 orbit 9can you see it?) for a small band of values of λ. This attracting period-3 orbit eventually becomes a repeller, and starts another period doubling cascade (period-6 to period-12, etc.). In fact, there exists a period doubling cascade within this diagram for each prime number n. Look carefully and check in the book in chapter 11 for more details on this fascinatingly simple complicated map.