MATH 421 DYNAMICS

Week 11 Lecture 2 Notes

1. Topological Entropy

Recall our indicators of dynamical complexity from before: topological transitivity, minimality, density of periodic orbits, chaos, growth rates of periodic orbits, etc. These properties are called *dynamical invariants* under conjugacy:

- Allows you to study a new dynamical system by establishing a conjugacy with a known one.
- Allows you to classify dynamical systems (everything conjugate to a known dynamical system has the same dynamical invariants.

Theorem 1. *Any two degree-2, expanding maps of S^1 are conjugate.*

Thus, the only degree-2 expanding map to study is the map $E_2 : S^1 \to S^1$, $E_2(x) = 2x \mod 1$.

Example 2. Rotation number classifies circle homeomorphisms

Here is a new dynamical invariant:

Definition 3. The *topological entropy* of a map is the exponential growth rate of the number of orbit segments distinguishable with arbitrary precision,

Note: This is the analytical version of what are called Lyapunov exponents.

Definition 4. *Lyapunov Exponents* are numbers which represent the exponential rate of divergence of nearby trajectories.

The idea here is to take two trajectories of initial separation $\delta_0 > 0$. If after time-t, the separation is $\delta_t = e^{\lambda t}\delta_0$, then the Lyapunov exponent is λ. Note that this depends largely on the direction of the measurement. Different directions of travel will result in different separation rates. Of interest typically is the largest:

- For C^1-dynamical systems, the exponents are related directly to the eigenvalues of the Jacobian matrix, a local linearization of the system.
- for C^0-systems, there is no Jacobian matrix to work with. However, one can still calculate the maximum exponent via
$$\lambda = \lim_{t\to 0} \frac{1}{t} \lim_{\delta_0 \to 0} \log \frac{\delta_t}{\delta_0}.$$
- Calculations of Lyapunov exponents are usually done numerically and only locally. Only rarely can they be calculated analytically or over the entire space.

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Now to actually define topological entropy, we will need some more machinery:

1.1. **Capacity.** Let X be a compact metric space (both closed and bounded). A set $E \subset X$ is called *r-dense* if, using the metric,

$$X \subset \bigcup_{x \in E} B_r(x).$$

That is, if X can be covered by a set of r-balls all of whose centers lie in E. Then the *r-capacity* of X, with metric d is the minimal cardinality of any r-dense set. Denote the r-capacity of a set X by $S_{X,d}(r)$ (or simply $S_d(r)$ when the space X is either understood or not necessary to be explicit about).

Some Notes:

- This is simply a way of denoting the "thickness" of sets which have no actual volume by how they sit inside X (think cantor sets sitting inside an interval).
- It does not really matter ultimately, but we will mostly consider closed balls in these calculations.
- Some examples:

  Example 5. $\mathbb{Z}$ is r-dense in $\mathbb{R}$ if $r > \frac{1}{2}$ if the balls are open, and $r \geq \frac{1}{2}$ is the balls are closed.

  Example 6. $\mathbb{Z}^2$ is r-dense in $\mathbb{R}^2$ if $r > \frac{\sqrt{2}}{2}$ if the balls are open, and $r \geq \frac{\sqrt{2}}{2}$ is the balls are closed. Can you visualize this?

  Example 7. Let $I = [0,1]$ be the unit interval. Using open balls here, the $\frac{1}{2}$-capacity of I is 2. The $\frac{1}{4}$-capacity is 3. The $\frac{1}{8}$-capacity is 5, and the $\frac{1}{16}$-capacity is 9. One can show that $S_d\left(\frac{1}{2^n}\right) = 2^{n-1} + 1$.

  Exercise 1. Show this.

  Exercise 2. Determine a bound on r for which $\mathbb{Z}^3$ is r-dense in $\mathbb{R}^3$.

- These calculations work well with Cantor Sets. Studying how $S_d(r)$ changes as r changes (really, it is the order of magnitude of $S_d(r)$) leads to a generalized notion of dimension.

2. Box Dimension

A rough notion of dimension for a topological space would be how many coordinates it would take to completely determine a point in the space (in relation to the other points). For example, the common description of the two-sphere S^2 is as the unit-sphere in $\mathbb{R}^3$; the set of all unit-length vectors in $\mathbb{R}^3$. However, using spherical coordinates (ρ, θ, ϕ) (see the connection), all of these points have coordinate $\rho = 1$, and hence each point on the sphere only requires two coordinates to differentiate between them. Hence, in a way, S^2 is two-dimensional as a space. This notion is not mathematically precise, however, as there do exist curves (1-dimensional lines) that can "fill" a two-dimensional space (Peano curves, some examples are called). Hence is this curve 1-dimensional, or 2 dimensional? Here, we will explore one mathematically precise notion of dimension (there are many), which will be useful in our definition of topological entropy.

Definition 8. A metric space X is called *totally bounded* if $\forall r > 0$, X can be covered by a finite set of r-balls all of whose centers are in X.

Really, this definition is technical, and is meant to account for the general metric space aspect of this discussion. That the centers need to be within X really only is a factor when the metric space X is a subspace of another space Y (otherwise there is no "outside" of X. And in Euclidean space, the notion of totally bounded is just the common notion of bounded that you are used to.

Definition 9. For X totally bounded,

$$\mathrm{bdim}(X) := \lim_{r\to 0} \frac{-\log S_{(X,d)}(r)}{\log r}$$

is called the *box dimension* of X.

Notes:

- This concept is also called the Minkowski-Bouligard dimension, or the entropy dimension or the Kolmogorov dimension.
- This is an example of the idea of fractional dimension; some sets may look bigger then 0-dimensional, yet smaller than 1-dimensional, for example.
- In the case where this limit may not exist (I cannot think of an example where it wouldn't for a totally bounded set), certainly one can use the limit superior or the limit inferior to gain insight as to the "size" of a set.
- To calculate, really simply find a sequence of r-sizes going to 0, and calculate the r-capacities for this sequence. If the limit exists, then ANY sequence of r's going to 0, with their associated r-capacities will determine the same box dimension (Why?).

Example 10. Calculate $\mathrm{bdim}(I)$, for $I[0,1]$ with the metric d that I inherits from $\mathbb{R}$. Recall that if we were to use closed balls, then the $\frac{1}{2^n}$-capacity for I is $S_{(X,d)}\left(\frac{1}{2^n}\right) = 2^{n-1}$. But for open balls, we have $S_{(X,d)}\left(\frac{1}{2^n}\right) = 2^{n-1} + 1$. The box dimension should be the same for both. Indeed, it is: For the harder one,

$$\begin{aligned}\mathrm{bdim}\,(I) &= \lim_{r\to 0} \frac{-\log S_{(X,d)}(r)}{\log r} = \lim_{n\to\infty} \frac{-\log\left(2^{n-1}+1\right)}{\log\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{\log\left(2^{n-1}+1\right)}{\log 2^n} \\ &\geq \lim_{n\to\infty} \frac{\log 2^{n-1}}{\log 2^n} = \lim_{n\to\infty} \frac{n-1}{n} = 1,\end{aligned}$$

and

$$\begin{aligned}
\operatorname{bdim}(I) &= \lim_{r\to 0} \frac{-\log S_{(X,d)}(r)}{\log r} = \lim_{n\to\infty} \frac{-\log\left(2^{n-1}+1\right)}{\log\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{\log\left(2^{n-1}+1\right)}{\log 2^n} \\
&\le \lim_{n\to\infty} \frac{\log 2^{n-1}\cdot n}{\log 2^n} = \lim_{n\to\infty} \frac{\log 2^{n-1}}{\log 2^n} + \lim_{n\to\infty} \frac{\log n}{\log 2^n} \\
&= \lim_{n\to\infty} \frac{n-1}{n} + \lim_{n\to\infty} \frac{\log n}{n} = 1 + 0 = 1.
\end{aligned}$$

Hence $\operatorname{bdim}(I) = 1$. Using the closed ball construction is even easier.

Example 11. Let C be the Ternary Cantor Set. Show $\operatorname{bdim}(C) = \frac{\log 2}{\log 3}$. Here, assume that C sits inside I from the previous example, and again inherits its metric d from I. And since we can choose our sequence of r's going to zero, we will choose $r = \frac{1}{3^n}$, and consider only closed balls. Then one can show that $S_{(C,d)}\left(\frac{1}{3^n}\right) = 2^{n+1}$. (Think about this: At each stage, we remove the middle third of the remaining intervals. That means that at each stage we can cover each interval by a closed ball of radius $\frac{1}{3^n}$. But the mid-point is NOT in C, Hence we have to shift over a bit to find a point in C. Which means that we will need another ball to cover the remainder on this side. This over covers the interval, but is not enough to cover two adjacent intervals. And since at each stage there are 2^{n+1} intervals, we are done. See the figure.

The calculation is now easy:

$$\begin{aligned}
\operatorname{bdim}(C) &= \lim_{r\to 0} \frac{-\log S_{(C,d)}(r)}{\log r} = \lim_{n\to\infty} \frac{-\log\left(2^{n+1}\right)}{\log\left(\frac{1}{3^n}\right)} = \lim_{n\to\infty} \frac{\log\left(2^{n+1}\right)}{\log 3^n} \\
&= \lim_{n\to\infty} \frac{n+1}{n}\cdot\frac{\log 2}{\log 3} = \frac{\log 2}{\log 3}.
\end{aligned}$$

Exercise 3. Let $B = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$. Calculate $\operatorname{bdim}(B)$.

In fact, we have the following:

Theorem 12. *Let $C \subset I$ be the Cantor set formed by removing the middle interval of relative length $1 - \frac{2}{\alpha}$ at each stage. Then*

$$\mathit{bdim}(C) = \frac{\log 2}{\log \alpha}.$$

A special note: All Cantor sets are homeomorphic. Yet, if we change the size of a removed interval at each stage, we effectively change the box dimension. This means that box dimension is NOT a topological invariant (remains the same under topological equivalence). Since a homeomorphism here would also act as a conjugacy between two dynamical systems on Cantor Sets, this also means that box dimension is also NOT a dynamical invariant.

For $f : X \to X$, a continuous map on a metric space (X, d), consider a sequence of new metrics on X indexed by $n \in \mathbb{N}$:

$$d_n^f(x, y) := \max_{0 \le i \le n-1} d\left(f^i(x), f^i(y)\right).$$

Here, the new metrics d_n^f actually measure a "distance" between orbit segments

$$\begin{aligned} \mathcal{O}_{x,n} &= \left\{x, f(x), \ldots, f^{n-1}(x)\right) \\ \mathcal{O}_{y,n} &= \left\{y, f(y), \ldots, f^{n-1}(y)\right) \end{aligned}$$

as the farthest that these two sets diverge along the orbit segment, and assigns this distance to the pair x and y.

Exercise 4. Show for a given n that d_n^f actually defines a metric on X.

Now, using the metric d_n^f, we can define an r-ball as the set of all neighbor points y whose nth orbit-segment $\mathcal{O}_{y,n}$ stays within r distance of $\mathcal{O}_{x,n}$:

$$B_r(x, n, f) = \left\{ y \in X \,\middle|\, d_n^f(x, y) < r \right\}.$$

Convince yourself that as we increase n, the orbit segment is getting longer, and more and more neighbors y will have orbit segments that move away from $\mathcal{O}_{x,n}$. Thus the r-ball will get smaller as n increases. But by continuity, the r-balls for any n will always be open sets in X that have x as an interior point. Also, as r goes to 0, the r-balls will also get smaller, right?

Now define the r-capacity of X, using the metric d_n^f and the new r-balls $B_r(x, n, f)$, denoted $S_{(X,d)}(r, n, f)$ (this is the SAME notion of r-capacity as the one we used for the box dimension! We are only changing the metric on X to d_n^f. But the actual calculations of the r-capacity depend on the choice of metric). As before, as r goes to 0, the r-balls shrink, and hence the r-capacity grows. And also, as n goes to ∞, we use the different d_n^f to measure ultimately the distances between entire positive orbits. This also forces the r-balls to shrink, and hence the r-capacity to grow. What is the exponential growth rate of the r-capacity as $r \to 0$? This is the notion of topological entropy:

Definition 13. Let $h_d(f, r) := \overline{\lim}_{n\to\infty} \dfrac{\log S_d(r, n, f)}{n}$. Then

$$h_d(f) := \lim_{r\to 0} h_d(f, r)$$

is called the *topological entropy* of the map f on X.

Next time, we will expound on this, and do a calculation to "see" the structure of this construction.