MATH 421 DYNAMICS

Week 11 Lecture 1 Notes

Proposition 1. *$F_L : \mathbb{T}^2 \to \mathbb{T}^2$, the linear hyperbolic automorphism of the two torus given by the hyperbolic matrix L is topologically mixing.*

Corollary 2. *F_L is chaotic.*

For a brief idea why the previous proposition is true, recall for F_L given by the matrix $L = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, the eigenvalues were $\lambda = \frac{3 \pm \sqrt{5}}{2}$, and the eigenvalue greater than 1 (the "expending" eigenvalue) has eigendirection given by the vector $\begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix}$. Choose a small open line segment T along the line $y = \left(\frac{-1+\sqrt{5}}{2}\right) x + c$ within the box representing the torus. As we iterate the map, the orientation of the line stays the same, while the length of the line grows by a factor of $\lambda = \frac{3+\sqrt{5}}{2}$ at each iterate. For $N >> 1$, we would find that the length of $F_L(T)$ will be huge, and wrap around the torus quite densely. In fact, we can choose this N so that $F_L^N(T)$ will intersect ANY ball of radius ϵ in $\mathbb{T}^2$. Hence choose any ϵ-ball V and any other ϵ-ball U, and take as our T the diameter of U in the direction of the line $y = \left(-1 + \sqrt{5}2\right) x + c$. Then after the above chosen N, we would have $F_L^n(U) \cap V \neq \varnothing$, for all $n > N$. Hence F_L is topologically mixing on $\mathbb{T}^2$.

And finally, we will not prove this explicitly, but we have the following:

Proposition 3. *The logistic map $f_\lambda : C \to C$, where $\lambda > 2 + \sqrt{5} > 4$, and where $C \in [0,1]$ is the Canter set of points whose entire orbits stay within $[0,1]$ is expanding.*

Proposition 4. *f_λ as above, is topologically mixing.*

Corollary 5. *f_λ as above, is chaotic on C.*

1. Sensitive Dependence on Initial Conditions

So the next questions is: What information does chaos, as a property, convey about the dynamical system? Flippantly speaking, it tells us that the orbit structure is quite complicated. It tells us that arbitrarily close to a periodic point, are non-periodic points whose orbits are dense in the space. On the other hand, it tells us that arbitrarily close to a point whose orbit is dense in the space, are periodic points of arbitrarily high period. Hence simply being very close to a point of a certain type does not mean that the orbits will be similar. This means that one cannot rely on estimates or precision to help determine orbit behavior. Mathematically, it means the following:

Definition 6. A map $f : X \to X$ of a metric space is said to exhibit a *sensitive dependence on initial conditions* if $\exists \Delta > 0$ (called a sensitivity constant), where $\forall x \in X$ and $\forall \epsilon > 0$, $\exists$ a point $y \in X$ where $d(x,y) < \epsilon$ and $dA(f^N(x), f^N(y)) > \Delta$ for some $N \in \mathbb{N}$.

Date: April 16, 2013.

There are lots of notes to say on this topic:

- The idea here is, for certain constants, no matter how small a neighborhood of a chosen point x you start, there will always be a point y in this neighborhood that after a time, its neighborhood will be far away from the orbit of x.
- The existence of at least one point in each neighborhood of an arbitrary point x whose orbit veers away from the orbit of x is the notion that everywhere there is an expanding direction (think of a differentiable map whose derivative everywhere, as a matrix, has at least one eigenvalue of modulus greater than 1). This is like the hyperbolic action on the torus.
- This idea was quite profound: Early developers of classical mechanics tended to believe that eventually we would understand the universe completely. Given the universes state in an instant, we should be able to predict its state at any future moment. This was the thinking around the early 1800's of people like Laplace.
- Poincare, in the late 1800's, saw this phenomenon of a sensitive dependence on initial conditions in the classical three-body problem. He understood immediately that the earlier reasoning was flawed. Indeed, knowing the precise state of all things in the universe was impossible. And with the presence of a sensitive dependence on initial conditions (even in the simplistic three-body problem), a reasonable approximation to the universe's state in an instant would never be good enough to make good long term predictions.
- Edward Lorentz, studying early climate models on a computeris like at MIT in the 1960's, saw his deterministic (though nonlinear!) computer model make wildly divergent predictions given the exact same input values in redundant runs of his program. Puzzled as to why this was the case, it became clear that the model was fine. It was the assumption that any number is known to infinite precision in a computer. For example, zero is not zero on a computer. Setting a variable to 0 on a computer makes the number 0 only to within a certain precision (it stores the number in a certain number of bytes). For example, 0 in single precision, is only 0 down to 10^-7. If in the model this number is multiplied by a very large number, any variance from true 0 would be multiplied into the realm where it will change the calculations. Do this same run twice and you would get two different values for the result. This small variance is like trying to grab a point like x above and instead getting a nearby number like y instead. In the calculations, the resulting orbits would veer away from each other, and the results would be different. Eventually, this was the discovery that Lorentz had made. Incidentally, the Lorentz Butterfly is an example of what is called a "strange attractor", and came out of the puzzle Lorentz created.
- Isometries cannot exhibit a sensitive dependence on initial conditions. Why not?
- For $f(x) = 2x \mod 1$, distance between nearby points grow exponentially by 2^n. This is a sensitive dependence on initial conditions. Eventually, this distance is larger than 1, and at this point, future iterates of each orbit tend to look unrelated to each other.

- for a map exhibiting a sensitive dependence on initial conditions in a compact space (closed and bounded), one can see how the orbit structure can be complicated. If all orbits are moving away from each other, and yet cannot go beyond the boundaries of the space, they just wind up mixing around each other. Think of smoke rising form a hot cup of coffee, or rising from a cigarette, and you can see just how complicated the orbits can be in this case.

Theorem 7. *Chaotic maps exhibit a sensitive dependence on initial conditions, except when the entire space consists of one periodic orbit.*

Example 8. Let

$$X = \left\{ 0\,,\ \frac{1}{5}\,,\ \frac{2}{5}\,,\ \frac{3}{5}\,,\ \frac{4}{5} \right\},$$

and $f : X \to X$, $f(x) = x + \frac{1}{5} \mod 1$. Here, f is continuous with respect to the topology X inherits from $\mathbb{R}$ (this is called the subspace topology: Write $X \subset \mathbb{R}$ in the obvious way. Then declare any subset of X to be open if it can be written as an intersection of X with open set of $\mathbb{R}$. This is really the trivial topology of X as it is a finite, discrete subset of $\mathbb{R}$, so each point of X is open). Here, X certainly has a dense orbit (every point of X live in the orbit of $\frac{1}{5}$). And the set of all periodic points of X are dense in X (ALL points of X are periodic). Hence f is chaotic on X. But there certainly is not a sensitive dependence on initial conditions here.

Example 9. The twist map on the cylinder does have a sensitive dependence on initial conditions. To see this, recall that each horizontal circle is invariant, and has a different rotation along it which is a linear function of height. Now take any point x, and any small neighborhood of x. This small neighborhood will include points on horizontal circles different from that of x. Choose any one of these points. Eventually, x and this other point will wind up pretty much on opposite sides of the cylinder. So what is the sensitivity constant (the largest such Δ)?

Exercise 1. For the twist map and the standard parameterization (and metric) of the circle given by the exponential map $f(x) = e^{2\pi i x}$, show the sensitivity constant is $\frac{1}{2}$.

Proposition 10. *A topological mixing map (on a non-trivial space) exhibits a sensitive dependence on initial conditions.*

Remark 11. Perhaps a better definition of chaos is one which requires a sensitive dependence on initial conditions as a third condition along with the other two. This would discount the "chaotic" map in the above example (which is hardly chaotic in a non-mathematical sense), while not restricting in any detrimental way the intent of the property. In fact, this is a fairly widely accepted set of conditions for a map to be chaotic. Note also that the condition of sensitive dependence on initial conditions cannot by itself constitute a chaotic system. The twist map is an example of a system hardly in a chaotic state. And even the *star node*, the equilibrium at the origin of the map $\dot{\mathbf{x}} = I_2\mathbf{x}$ exhibits a sensitive dependence on initial conditions. Again, hardly chaotic, with neither of the other two conditions satisfied.

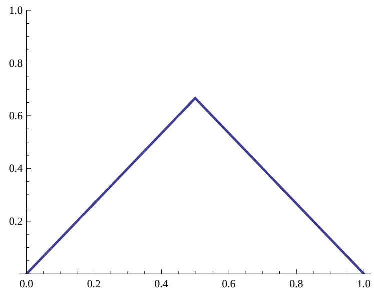
2. Topological Conjugacy

Definition 12. Suppose $g : X \to X$ and $f : Y \to Y$ are maps of metric spaces and there exists a surjective map $h : X \to Y$ such that

$$h \circ g = f \circ h.$$

Then f is called a factor of g under h and f is said to be *topologically semiconjugate* to g via the *semiconjugacy* h. Furthermore, if h is a homeomorphism, then h is a conjugacy and f is topologically conjugate to g. We say in this case that $f \sim_h g$.

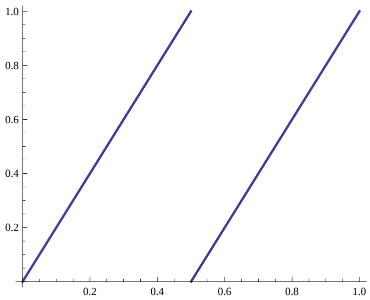
Note: In a (semi)conjugacy, orbits are taken to orbits via h. Thus the orbit structure of g and that of f are the same. It is for this that in dynamical systems, the notion of conjugacy is the notion of equivalence, or isomorphism. As we will see, the existence of a conjugacy allows us to study hard-to-study dynamical systems by instead establishing a conjugacy between them and easy to study ones.



The *tent map* $T_r : [0,1] \to [0,1]$ (at right) is a continuous, piece-wise linear, unimodular interval map given by

$$T_r(x) = \begin{cases} rx & \text{if } 0 \le x \le \frac{1}{2} \\ r(1-x) & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

This is also sometimes called the sawtooth function. Its height, at $x = \frac{1}{2}$, is obviously $\frac{r}{2}$.

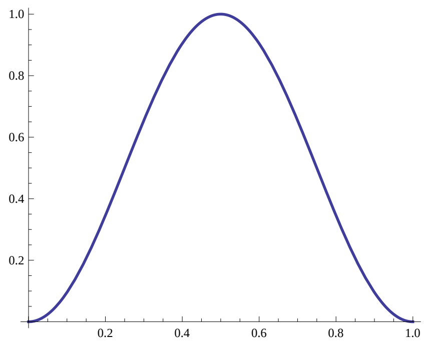


In contrast, the linear expanding map E_2 on S^1 has the graph at left. As a map on S^1, it is certainly continuous (here, the point 0 is the same as 1 in both the domain and the range. Hence the map can run off the top of the graph and reappear at the bottom and still be continuous). As a graph in the unit square displays much of the same information as the tent map when the peak is precisely at 1. In fact, we can define E_2 as an interval map via

$$E_2(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Proposition 13. *The logistic map $f_4(x) = 4x(1-x)$ on $[0,1]$ is topologically semi-conjugate to $E_2(x) = 2x \mod 1$ on S^1 via $h_1(x) = \sin^2 \pi x$, and topologically conjugate to the tent map $T_2 : [0,1] \to [0,1]$ via the conjugacy $h_2(x) = \sin^2 \frac{\pi}{2} x$.*

First, we will state without much detail that $h_2(x)$ is a homeomorphism. It is continuous, 1-1 and onto $[0,1]$, and



its inverse $h_2^{-1}(x) = \frac{2}{\pi}\arcsin\sqrt{x}$ is also continuous on $[0,1]$. Instead, the map $h_1 : S^1 \to I$ is a 2-1 map, and hence does not have an inverse. Thus $h_1(x)$ is not a homeomorphism (which makes sense, since S^1 is not homeomorphic to I). $h_1(x)$ is surjective, however, as one can see here.

Proof. Here, we will explicitly show the conjugacies. First, we show $h_1 \circ E_2 = f_4 \circ h_1$. This semi-conjugacy condition needs to be parsed along the linear pieces of E_2. Hence we want

$$h_1(2x) = f_4\left(\sin^2 \pi x\right) \text{ for } 0 \leq x \leq \frac{1}{2} \text{ and} \tag{1}$$

$$h_1(2x-1) = f_4\left(\sin^2 \pi x\right) \text{ for } \frac{1}{2} \leq x \leq 1. \tag{2}$$

As for the left hand sides of these two equations, in Equation **??**, we get $h_1(2x) = \sin^2 \pi(2x) = \sin^2 2\pi x$.. And in Equation **??**, on the left, we also have

$$h_1(2x-1) = \sin^2 \pi(2x-1) = \sin^2(2\pi x - \pi) = \sin^2 2\pi x$$

since $\sin(x-\pi) = -\sin x$. On the right hand side of each, we see

$$\begin{aligned}
f_4\left(\sin^2 \pi x\right) &= 4\left(\sin^2 \pi x\right)\left(1 - \sin^2 \pi x\right) \\
&= 4\left(\sin^2 \pi x\right)\left(\cos^2 \pi x\right) \\
&= 4\left(\frac{1}{2} - \frac{1}{2}\cos 2\pi x\right)\left(\frac{1}{2} + \frac{1}{2}\cos 2\pi x\right) \\
&= 4\left(\frac{1}{4} - \frac{1}{4}\cos^2 2\pi x\right) \\
&= 4\left(\frac{1}{4}\sin^2 2\pi x\right) = \sin^2 2\pi x.
\end{aligned}$$

As for the conjugacy $h_2(x)$, we need to show that $h_2 \circ T_2 = f_4 \circ h_2$. Again, we would need to parse this condition along the two linear pieces of T_2. The two resulting equations are almost identical to the previous case. In fact, Equation **??** is precisely the same with all of the factors π replaced by $\frac{\pi}{2}$ (thereby replacing h_1 with h_2). And for Equation **??**, this time we get

$$h_2(2-2x) = \sin^2 \frac{\pi}{2}(2-2x) = \sin^2 \pi(1-x) = \sin^2 \pi - \pi x = \sin^2 2\pi x$$

since $\sin(\pi - x) = \sin x$. $\square$

Notes:

- The maps h_1 and h_2 are truly related, and come from the relationship between S^1 and I. Conjugacies are really all about maps that take orbits to orbits, and any map that satisfies this condition will transfer the dynamics of one system to the other. In this case, both the tent map and the expanding circle map have a certain symmetry about them; $E_2\left(x+\frac{1}{2}\right) = E_2(x)$ on I, while $T_2(x) = T_2(1-x)$. f_4 shares the latter property with T_2, and $T_2(x) = 1 - E_2(x)$ on the interval $\frac{1}{2} \le x \le 1$. The sine function has the appropriate property that $\sin \pi x = \sin \pi(1-x)$. The sine function is also a beautiful way to map S^1 down onto an interval. Indeed, view points of S^1 as $e^{2\pi i x}$, for $x \in I$, and the real part of $z = e^{2\pi i x} \in S^1$ is $\cos 2\pi x$. We can scale this as a "tent-like" map on I as the function
$$x \mapsto \frac{1-\cos 2\pi x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2\pi x = \sin^2 \pi x.$$
This is precisely h_1 above. For h_2, halving the angle makes h_2 1-1 on I.
- Once a (semi)-conjugacy is specified, ALL of the interesting dynamics of the logistic map for $\lambda = 4$ are present in the tent map for $r = 2$, as well as the linear expanding map E_2 on S^1.

Next class, we will explore the other two examples in our recent models.