MATH 421 DYNAMICS

Week 10 Lectures Notes

Question 1. *How are any periodic points distributed?*

we have the following proposition:

Proposition 2. *The set of all periodic points of $F_L : \mathbb{T}^2 \to \mathbb{T}^2$ is dense in $\mathbb{T}^2$ and $P_n(F_L) = \lambda_1^n + \lambda_1^{-n} - 2$.*

Proof. The first claim we will make to prove this result is the following: *Every rational point in $\mathbb{T}^2$ is periodic.* To see this, note that every rational point in $\mathbb{T}^2$ is a point in the unit square with coordinates $x = \frac{s}{q}$, and $y = \frac{t}{q}$, for some $q, s, t \in \mathbb{Z}$. For every point like this, $F_L(x, y)$ is also rational with the same denominator (neglecting simplification, do you see why?) But there are only q^2 distinct points in $\mathbb{T}^2$ which are rational and which have q as the common denominator. Hence, at some point, $\mathcal{O}_{(x,y)}$ will start repeating itself. Hence this claim is proved. Now notice that the set of all rational points in $\mathbb{T}^2$ is dense in $\mathbb{T}^2$, or

$$\overline{\mathbb{Q} \cap [0,1]} \times \overline{\mathbb{Q} \cap [0,1]} = [0,1]^2.$$

Hence the periodic points are dense in $\mathbb{T}^2$.

The next claim is: *Only rational point are periodic.* To see this, assume $F_L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$F_L^n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \mod 1,$$

and this forces the system of equations

$$\begin{array}{c} ax + by = x + k \\ cx + dy = y + \ell \end{array}, \text{ for } k, l \in \mathbb{Z}.$$

Simply solve this system for x and y and you get that $x, y \in \mathbb{Q}$.

Exercise 1. Solve this system for x and y.

The number of periodic points can be found by creating a new linear map. Define

$$G_n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = F_L^n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) - \begin{bmatrix} x \\ y \end{bmatrix} = (F_L^n - I_2)\begin{bmatrix} x \\ y \end{bmatrix}.$$

The n-periodic points are precisely the kernel of this linear map:

$$P_n(F_L) = \ker(G_n) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; G_n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}.$$

We can easily count these now. They are precisely the pre-images of integer vectors!

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Claim. All pre-images of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under the map $G_n = F_L^n - I_2$ are given by $\mathbb{Z}^2 \cap (L^n - I_2)\left([0,1) \times [0,1)\right)$.

- Since F_L is to be understood as simply the matrix L where images are taken modulo 1, the map G_n is simply the map $L^n - I_2$ where images are taken modulo 1. Hence we can study the effect of G_n by looking at the image of $L^n - I_2$.
- To avoid over-counting points, we modify our unit square, eliminating twice-counted points (on the edges) and quadruply-counted points (the corners). Consider the "half-open box" $[0,1)^2$ as our model of $\mathbb{T}^2$. In this model, every point lives in its own equivalence class.

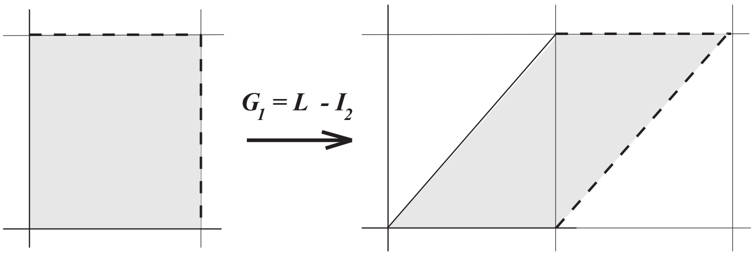
We try a few early iterates:

Example 3. $G_1 = L - I_2$. Here

$$G_1 = L - I_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

This map is a shear on $[0,1)^2$ and we see that the only integer vector in the image is the origin. Thus

$$P_1(F_L) = \lambda^1 + \lambda^{-1} - 2 = \frac{3+\sqrt{5}}{2} + \frac{3-\sqrt{5}}{2} - 2 = 3 - 2 = 1.$$



Example 4. $G_2 = L^2 - I_2$. Here

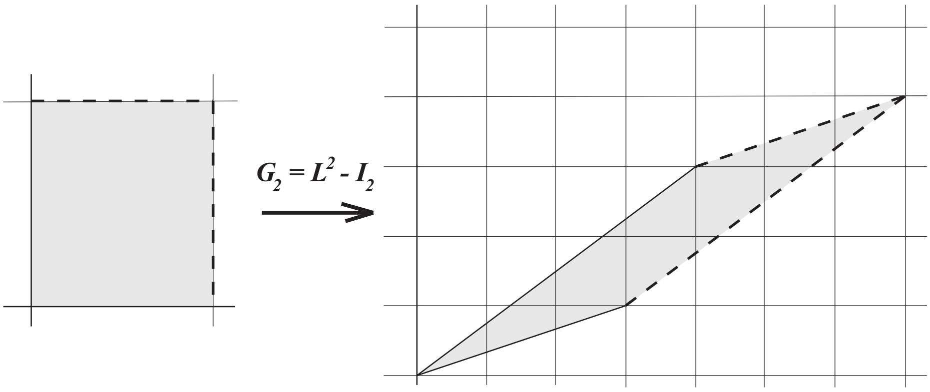
$$G_2 = L^2 - I_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}.$$

This map is a little more complicated, and we see that there are a few more integer vectors in the image, namely the points $(2,1)$, $(3,2)$, $(4,2)$, and $(5,3)$. And since

$$P_2(F_L) = \lambda^2 + \lambda^{-2} - 2 = 7 - 2 = 5,$$

we see that the formula continues to hold (where is the fifth point?)

Exercise 2. What were the original points in $[0,1)^2$ that correspond to these 5 integer vectors under G_2?



This proof ends by establishing an interesting geometric link: The area of $G_n\left([0,1)^2\right)$ is precisely equal the number of integer-vectors in the image. And the latter is given by

$$\det\left(G_n\right) = \lambda^n + \lambda^{-n} - 2,$$

where λ is the largest eigenvalue (in magnitude) of L. $\square$

Note: G_2 on $\mathbb{R}^2$ is NOT area preserving! In fact,

$$\det(G_2) = \begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} = 5.$$

Note that the map F_L above was area-preserving on the torus. It is also invertible (any determinant one matrix with integer coefficients is invertible, and the inverse is also of determinant one with integer entries!) However, area-preserving does NOT ensure invertibility of the map. The prime example is the circle map $E_m : S^1 \to S^1$, where $E_m(z) = z^n$. The map is area-preserving, if we sum all of the lengths of the disjoint pre-images of small sets. But it is also of degree m. And if $|m| > 1$, the map is m to 1. Invertibility is a very desirable quality for a map, as it allows us to work both forwards and backwards in constructing orbits. Fortunately, there are ways to study non-invertible maps by encoding their information in a (different) invertible dynamical system. We will introduce this concept here, but not spend a lot of time on it for now:

0.1. **Inverse Limits.** Let $f : X \to X$ be continuous on a metric space X, and $x \in X$. When f is invertible, constructing the backwards orbit amounts to constructing the inverse of the map, and

$$\mathcal{O}_x = \left\{ x \in X \;\middle|\; f^n(x) = y,\ n \in \mathbb{Z} \right\}.$$

When f is not invertible, one would usually have to make a choice to go backwards, making the inverse map not well-defined. There is a fix to this: Encode all possible orbits of points in X as a set of integer-indexed sequences, where to each point $x \in X$, one associates its forward orbit starting at x at the 0th place, $\{x_n\}_{n\in\mathbb{N}}$ and for each choice of preimage of x, one constructs a separate sequence by pre-adjoining each preimage of x. These new sequences now start at the -1st place, and look like $\{x_n\}_{n=-1}^{\infty}$. Continue adjoining preimages to each of the new sequences until you have a large set of sequences, all indexed by Z, or $\{x_n\}_{n\in\mathbb{Z}}$. This large set of sequences can be made into a topological space (actually a metric space) with a bit of cleverness, so that sequences which are almost the same are considered “close”.

Example 5. For $E_2(x) = 2x \mod 1$ on S^1, some of these sequences which correspond to $x_0 = 1$ look like

$$\begin{array}{c} 0^{\text{th}} \text{ place} \\ \downarrow \end{array}$$

$$\left\{\dots,\frac{1}{8},\frac{1}{4},\frac{1}{2},\ \mathbf{1}\ ,1,1,\dots\right\}$$
$$\left\{\dots,\frac{1}{4},\frac{1}{2},1,\ \mathbf{1}\ ,1,1,\dots\right\}$$
$$\left\{\dots,\frac{3}{8},\frac{3}{4},\frac{1}{2},\ \mathbf{1}\ ,1,1,\dots\right\}$$
$$\left\{\dots,\frac{7}{8},\frac{3}{4},\frac{1}{2},\ \mathbf{1}\ ,1,1,\dots\right\}$$

Definition 6. For X a metric space and $f : X \to X$ continuous, the *inverse limit* is defined on the space of sequences

$$X' = \left\{ \{x_n\}_{n\in\mathbb{Z}} \;\middle|\; x_n \in X,\ f(x_n) = x_{n+1},\ \forall n \in \mathbb{Z} \right\}$$

by $F\left(\{x_n\}_{n\in\mathbb{Z}}\right) = \{x_{n+1}\}_{n\in\mathbb{Z}}$.

This is a new dynamical system defined by the map F on the *inverse limit space* X'. Note that since this map takes entire sequences to sequences, it is 1-1, and hence we can go backwards. On sequences, this map is invertible, since the entire history of a point is already in the “point” (read: sequence).

Example 7. Back to the map E_2 on S^1, the limit space is

$$\mathbb{S} = \left\{ \{x_n\}_{n\in\mathbb{Z}} \;\middle|\; x_n \in S^1,\ E_2(x_n) = x_{n+1},\ \forall n \in \mathbb{Z} \right\}$$

with the map $F\left(\{x_n\}_{n\in\mathbb{Z}}\right) = \{2x_n \mod 1\}_{n\in\mathbb{Z}}$. The space $\mathbb{S}$ is called a solenoid, and a picture of it site on the cover of the book.

1. Chaos and Mixing

Recall a map $f : X \to X$ on a metric space is *topologically transitive* if there exists a dense orbit. Some examples that we looked at included the irrational rotations of S^1 and the irrational linear flows on the two torus $\mathbb{T}^2$. Note that these examples had no periodic points at all, and all orbits were dense. Contrast that with the idea some dynamical systems seemed to be full of periodic points. Think of rational rotations of S^1 and rational linear flows on $\mathbb{T}^2$. Again, on these examples, all points were periodic, and none of these maps are topologically transitive.

These properties seem to be mutually exclusive, and are when the dynamics are relatively simple to describe. However, for dynamical systems which possess both a dense supply of periodic orbits as well as a dense orbit, the dynamics can be labeled quite complex. How complex?

Definition 8. A continuous map $f : X \to X$ of a metric space is said to be *chaotic* if

- f is topologically transitive,
- $\overline{Per(f)} = X$.

Notes:

- There are many definitions of chaos floating around in this area, as efforts to finally pin down the concept continue. This definition really is one of the better universal definitions we have for the concept. That said, there is still a slight problem even with this definition. Check out the theorem and example on chaos in the next lecture.
- Either one of these properties without the other means that the dynamics are relatively simple to describe.

Some examples that we were recently playing with:

(1) Let $E_m : S^1 \to S^1$ be the linear expanding map of S^1, for $|m| > 1$.
(2) Let $f_\lambda : C \to C$ be the logistic map for $\lambda > 4$, restricted to the Cantor set of point whose orbit lies completely within the unit interval.
(3) Let $F_L : \mathbb{T}^2 \to \mathbb{T}^2$ be the linear hyperbolic automorphism of the two-torus given by the linear automorphism of the plane determined by the hyperbolic matrix L.

In the first and third cases, we showed that the periodic points are dense in the respective spaces. Hence the dynamical systems are chaotic if we can show there actually exists a dense orbit. The same holds for the Cantor map, although we did not actually show that the periodic points are dense. However, showing directly that there exists a dense orbit is not easy. We will instead construct a bit more machinery, and show that these maps possess some stronger properties that transitivity. In this way, we can study the maps in more detail, and gain some additional insight into the structure of these dynamical systems. To start:

Proposition 9. *Let X be a complete separable metric space with no isolated points. For $f : X \to X$ continuous, the following are equivalent:*

(1) f has a dense orbit and is topologically transitive,
(2) f has a dense positive semiorbit,
(3) if $U, V \subset X$ are open and nonempty, $\exists N \in \mathbb{Z}$ such that $f^N(U) \cap V \neq \emptyset$,
(4) if $U, V \subset X$ are open and nonempty, $\exists N \in \mathbb{N}$ such that $f^N(U) \cap V \neq \emptyset$.

Remark 10. Recall that a metric space is complete if all Cauchy sequences converge. And X is separable if there exists a countable dense subset. These properties, along with the "no isolated points" condition, are technical in nature and while necessary, should not keep you from well understanding how this proposition works on the nice spaces we are used to. So don't worry too much at this point about these technicalities.

Proof. Obviously $4 \Rightarrow 3$ and $2 \Rightarrow 1$. If we can show that $3 \Rightarrow 2$ and $1 \Rightarrow 4$, we would be done. We will not do this, however. The real point of this exposition is to understand the relationship between 1 and 3. To this end, we will prove the statement $1 \Rightarrow 3$.

Let f be topologically transitive, with a dense orbit given by $\mathcal{O}_x$, $x \in X$. Then for any choice of nonempty, open sets $U, V \subset X$, $\exists n, m \in \mathbb{Z}$ such that $f^n(x) \in U$, and $f^m(x) \in V$. If we suppose for a minute that $m > n$, then we would get that $f^{m-n}(U) \cap V \neq \emptyset$. In the case that f is invertible, this makes sense, since $f^{-n}(U)$ would be a neighborhood of x, and then $f^m(f^{-n}(U)) \cap V$ would be a neighborhood of $f^m(x)$ by continuity. However, this works even in the case where f is not invertible. Simply think of $f^{-n}(U)$ as being the inverse (set theoretic) image of U (the set of all things that go to U under f^n). See the picture. □

Corollary 11. *A continuous, open map f of a complete metric space is topologically transitive iff there does not exist two disjoint, open f-invariant sets.*

It will help in understanding this last statement to understand the notion of an open map. we will get to that in a minute. However, the idea in the previous Proposition is that finding a dense orbit is equivalent to the notion that the orbit of ANY open set in X must eventually intersect any other open set in X actually provides a method of discovery for dense orbits. A set $V \subset X$ is f-invariant, if $f(V) \subset V$. Now assume that you have such a set V which is open. Now take any other open set U. Whether it is invariant or not, its entire orbit $\mathcal{O}_U$ is a union of all of its images and is hence open in X. The Corollary says that if the map is topologically transitive, then $\mathcal{O}_U$ must intersect V. Put this way, the two notions look very much alike.

To better get some of these ideas, lets go over a bit of topologically:

Definition 12. A *topology* on a set X is a well-defined notion what constitutes an open subset of X.

What well-defined means is: A topology on X is a collection $\mathcal{T}_X$ of subsets of X that satisfy

- $\varnothing$ and X are in $\mathcal{T}_X$,
- the union of the elements of any subcollection of $\mathcal{T}_X$ is in $\mathcal{T}_X$, and
- the intersection of the elements in any finite subcollection of $\mathcal{T}_X$ is in $\mathcal{T}_X$.

For a topology $\mathcal{T}_X$ on X, the elements of $\mathcal{T}_X$ are called *open*. Also, any set that is given a topology, is called a *topological space*.

Example 13. The set of all open intervals $(a,b) \subset \mathbb{R}$ constitutes a topology on $\mathbb{R}$, called the *standard* topology $\mathcal{T}_{\mathbb{R}}$, for $-\infty \leq a \leq b \leq \infty$. It should be obvious that this allows all of $\mathbb{R}$ to be in $\mathcal{T}_{\mathbb{R}}$, and if we let $a = b$, then the element $(b,b) = \varnothing$ is also in $\mathcal{T}_{\mathbb{R}}$. The union of any collection of open intervals is certainly open also. Now, without the last condition, however, we would have a problem: Suppose we allowed that the intersection of any subcollection of $\mathcal{T}_{\mathbb{R}}$ to be in $\mathcal{T}_{\mathbb{R}}$. Then the set

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\}$$

would have to be open. But then all individual points would also be open, and thus by the middle constraint, any subset of X would be open! You can see why the third provision is necessary. Incidentally, there is a topology on $\mathbb{R}$ (or any other set), where each of the points is considered open. It is called the *trivial* topology on the set, and although it works via the definitions, it does not describe well the actual set as a space.

Some facts:

Definition 14. For $f : X \to Y$ a (not necessarily continuous) map between two topological spaces, f is *continuous* if whenever $V \subset Y$ is open (an element of its topology $\mathcal{T}_Y$), then $f^{-1}(V) \subset X$ is open (an element of $\mathcal{T}_X$).

This allows us to talk about maps being continuous between arbitrary topological spaces, in a way that is entirely compatible with what you already learned as the definition of continuity between spaces like subsets of $\mathbb{R}$ in Calculus I, or subsets of $\mathbb{R}^n$ in Calculus III. Then, we simply assumed the standard topologies on Euclidean space, and the notions of "nearness" which is at the center of continuity comes out of the little ϵ-balls used to define continuity.

Definition 15. $f : X \to Y$ is called an *open map*, if it is continuous and if whenever $U \subset X$ is open, then $f(U) \subset Y$ is open also.

While continuity is common among maps, “openness” is not, and is kind of a special property. When the map f has a continuous inverse, then f is open. But this is not that common a property.

Now, we know expanding maps of S^1 and hyperbolic automorphisms of $\mathbb{T}^2$ look messy dynamically. The question is: How messy are they?

Definition 16. A continuous map $f : X \to X$ is said to be *topologically mixing* if, for any two nonempty, open set $U, V \subset X$, $\exists N \in \mathbb{N}$, such that $f^n(U) \cap V \neq \varnothing$, $\forall n > N$.

Notes:

- Do you see how much stronger (more restricting) this is to topological transitivity? For instance, (topologically mixing)⇒(topologically transitive), but not vice-versa. To see why, think of the irrational rotations of the circle. The orbit of a small open interval will eventually intersect any other small open interval. But, depending on the rotation, will most likely leave again for a while before returning. This is not mixing!
- Actually, the problem with irrational circle rotations is a bit deeper; they are isometries:

  Lemma 17. *Isometries are not topologically mixing.*

  Proof. Under an isometry, the diameter of a set $U \subset X$, $diam(U)$ is preserved. Let $U = B_\delta(x) \subset X$ be a small δ-ball about a point $x \in X$. Here $diam(U) = 2\delta$ and $\forall n \in \mathbb{N}$, $diam(f^n(U)) = 2\delta$. Now choose $v_1, v_2 \subset X$, such that the distance between v_1 and v_2 is greater than 4δ. Let $V_1 = B_\delta(v_1)$ and $V_2 = B_\delta(v_2)$ (so that the minimal distance between these two balls is greater than 2δ). If we assume that the isometry $f : X \to X$ is top. mixing, then there will be a $k \in \mathbb{N}$, such that both $f^n(U) \cap V_1 \neq \varnothing$, and $f^n(U) \cap V_2 \neq \varnothing$. $\forall n > k$. But this is impossible since V_1 and V_2 are too far apart to both have nonempty intersection with an iterate of U. Hence f cannot be mixing. □

Proposition 18. *Expanding maps on S^1 are topologically mixing.*

Proof. for now, suppose that the expanding map is C^1. Differentiable expanding maps have the property that for $f : S^1 \to S^1$, $|f'(x)| \geq \lambda > 1$, $\forall x \in S^1$. Let $F : \mathbb{R} \to \mathbb{R}$ be a lift. it is an exercise to show that the lift also shares the derivative property, $|F'(x)| \geq \lambda$, $\forall x \in \mathbb{R}$. So choose a small closed interval $[a, b] \subset \mathbb{R}$, where $b > a$. Then, by the Mean Value Theorem, $\exists c \in (a, b)$, such that

$$|F(b) - F(a)| = |F'(c)||b - a| \geq \lambda(b - a).$$

Hence, the length of the iterate of the interval is greater by a factor of λ than the interval. This continues at each iterate of F, so that $\exists n \in \mathbb{N}$, such that $||F^n([a, b])|| > 1$. But then $\pi(F^n([a, b])) = S^1$.

Now simply grab the open interval (a, b), noting that $\pi((a, b))$ will also be open (on small intervals, π is a homeomorphism), and let $U = \pi((a, b))$. With V be any other open set in S^1, we are done. $\square$

Corollary 19. *Linear expanding maps of S^1 are chaotic.*