110.421 DYNAMICAL SYSTEMS

Contraction Map Example: Existence and Uniqueness criteria for first-order ODEs

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A good example of a contraction mapping and its utility is given by what are called Picard iterations. Consider the first order IVP

(1) $$\dot{y}(t) = f(t, y), \quad y(t_0) = y_0.$$

The question of whether Equation 1 has a solution, and when it has a solution, if it is uniquely defined, is a difficult one in general. However, due to the following theorem, the properties of $f(t, y)$ at and near the initial point (t_0, y_0) can ensure that unique solutions exist:

Theorem 1. *Suppose $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous in some rectangle*

$$R = \left\{ (t, y) \in \mathbb{R}^2 \mid \alpha < t < \beta, \gamma < y < \delta \right\},$$

containing the initial point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of Equation 1.

To give a good sense of why this is true, let's start with a definition:

Definition 2. An *operator* is a function whose domain and range are functions.

A good example of this is the derivative operator $\frac{d}{dx}$ which acts on all differentiable functions of one independent variable, and takes them to other (in this case, at least) continuous functions. Think

$$\frac{d}{dx}(x^2 + \sin x) = 2x + \cos x.$$

There are numerous technical difficulties in defining operators correctly, but for now, simply accept this general description.

We claim that any possible solution $y = \phi(t)$ (if it exists) to Equation 1 must satisfy

(2) $$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds$$

for all t in some interval containing t_0.

Exercise 1. Show that this is true (really, simply differentiate both sides to recover the ODE.)

At this point, existence of a solution to the ODE is *assured* in the case that $f(t, y)$ is continuous on R, as the integral will then exist at least on some smaller interval $t_0 - h < t < t_0 + h$ contained inside $\alpha < t < \beta$. Note the following:

- One reason a solution may not exist all the way out to the edge of R? What if the edge of R is an asymptote in the t variable?

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- A function does not have to be continuous to be integrable (step functions are one example of integrable functions that are not continuous. However, the integral of a step function IS continuous. And if we tried to place a step function into Equation 1, what comes out would not be a step function.

As for uniqueness, suppose $f(t, y)$ is continuous as above, and consider the following operator T, which take a function ψ to its image $T\psi$ defined by

$$T\psi = y_0 + \int_{t_0}^{t} f(s, \psi(s))\, ds.$$

We can stick in many functions for $\psi(t)$ and the image will be a different function $T\psi$ which is still a function of t (See the example at the end of this document). However, looking back at Equation 2, if we stick in the function $\phi(t)$ which solves our IVP, the image $T\phi$ should be the same as ϕ. In this case, we call such a function a *fixed point* of T, since $T\phi = \phi$.

Exercise 2. Find ALL fixed points for the derivative operator $\dfrac{d}{dx}$ on the domain $\mathbb{R}$.

Hence, instead of looking for solutions to the IVP, we can instead look for fixed points of the operator T, since any fixed point for T will also satisfy Equation 2 and hence solve the IVP. How do we do this? Fortunately, this operator has an interesting property. First, for T and operator and ϕ a function, define

$$T^n\phi = \overbrace{T(T(\cdots(T(\phi))\cdots))}^{n \text{ times}}.$$

Incidentally, this is called iterating the function T, and the above expression is called the nth iterate of ϕ under T.

Theorem 3. *Suppose you have a way to measure the distance between two functions $f(t)$ and $g(t)$ and call this distance $dist(f, g)$. If an operator T satisfies*

$$dist(Tf, Tg) \leq C \cdot dist(f, g), \quad \text{for some } 0 < C < 1,$$

then there is a single function ϕ that satisfies $T\phi = \phi$. In addition, this unique fixed point satisfies

$$\phi = \lim_{n\to\infty} T^n(g)$$

for any starting function $g(t)$.

Remark 4. Any operator that satisfies the distance criterion in this theorem is called a C-contraction, and in essence this theorem is the Contraction Principle, a common tool used in the study of ODEs and Dynamical Systems. We won't prove this theorem directly, but we will show by construction in the proof of Theorem 1 below that the operator T is a contraction.

Remark 5. Though not entirely necessary, it does make the proof easier to suppose that both $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are not only continuous on R, but bounded here also. This is because we can always slightly restrict R at an edge where one of the variables blows up. The proof is true even in this case. However, it is much easier to see with this restriction. As an example, let $f(t, y) = \log y$. Here, both f and $\frac{\partial f}{\partial y} = \frac{1}{y}$ are continuous on the rectangle $-1 < t < 1$, $0 < y < 1$. However, neither are bounded here. Create a new rectangle $\widetilde{R}$ by moving the left boundary of R slightly to the right; for a small $\epsilon > 0$, define $\widetilde{R}$ to be $-1 < t < 1$, $\epsilon < y < 1$. Here then both f and $\frac{\partial f}{\partial y}$ are continuous and bounded on $\widetilde{R}$.

proof of Theorem 1. Under the supposition that f and $\frac{\partial f}{\partial y}$ are bounded on R, call

$$M = \max_R \left| \frac{\partial f}{\partial y}(t, y) \right|,$$

and choose a small number $h = \frac{C}{M}$, where $C < 1$. Then define a distance within the set of continuous functions on the closed interval $I = [t_0 - h, t_0 + h]$ by

$$dist(g, h) = \max_{t \in I} \left| g(t) - h(t) \right|.$$

Then we have

$$\begin{aligned}
(3) \qquad dist(Tg, Th) &= \max_{t \in I} \left| Tg(t) - Th(t) \right| \\
(4) \qquad &= \max_{t \in I} \left| y_0 + \int_{t_0}^{t} f(s, g(s))\, ds - y_0 - \int_{t_0}^{t} f(s, h(s))\, ds \right| \\
(5) \qquad &= \max_{t \in I} \left| \int_{t_0}^{t} f(s, g(s)) - f(s, h(s))\, ds \right| \\
(6) \qquad &= \max_{t \in I} \left| \int_{t_0}^{t} \left[\int_{h(s)}^{g(s)} \frac{\partial f}{\partial y}(s, r)\, dr \right] \, ds \right| \\
(7) \qquad &\le \max_{t \in I} \left| \int_{t_0}^{t} M \left| g(s) - h(s) \right| \, ds \right| \\
(8) \qquad &\le \max_{t \in I} \int_{t_0}^{t} M \cdot dist(g, h)\, ds \\
(9) \qquad &\le \max_{t \in I} \left\{ M \cdot dist(g, h) \cdot |t - t_0| \right\}
\end{aligned}$$

Exercise 3. The justifications of going from Step 5 to Step 6 and from Step 6 to Step 7 are adaptations of major theorems from Calculus I-II to functions of more than one independent variable. Find what theorems these are and show that these are valid justifications. Can you see now why the continuity of $\frac{\partial f}{\partial y}(t, y)$ is a necessary hypothesis to the theorem?

Exercise 4. Justify why the remaining steps are true.

Now notice in the last inequality that since $I = [t_0 - h, t_0 + h]$, we have that

$$|t - t_0| \le h = \frac{C}{M}.$$

Hence

$$\begin{aligned}
dist(Tg, Th) &\le \max_{t \in I} \left\{ M \cdot dist(g, h) \cdot |t - t_0| \right\} \\
&\le M \cdot dist(g, h) \cdot \frac{C}{M} = C \cdot dist(g, h).
\end{aligned}$$

Hence T is a C-contraction and there is a unique fixed point ϕ (which is a solution to the original IVP) on the interval I. Here

$$\phi(t) = T\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

□

As an application, we can actually use this construction to "solve" an ODE:

Example 6. Solve the IVP

$$y' = 2t(1+y), \quad y(0) = 0.$$

Here, $f(t,y) = 2t(1+y)$, as well as $\frac{\partial f}{\partial y}(t,y) = 2t$ are both continuous on the whole plane $\mathbb{R}^2$. Hence unique solutions exist everywhere.

To actually find a solution, start with an initial guess to be

$$\phi_0(t) = 0.$$

Notice that this choice of $\phi_0(t)$ does not solve the ODE. But since the operator T is a contraction, iterating will lead us to a solution: Define $T\phi_0(t) = \phi_1(t)$, and similarly, define

$$\phi_n(t) = T\phi_{n-1}(t) = \overbrace{T(T(\cdots(T(\phi_0(t)))\cdots))}^{n \text{ times}}.$$

Here

$$\phi_1(t) = T\phi_0(t) = y_0 + \int_0^t 2s(1+\phi_0(s))\,ds = \int_0^t 2s(1+0)\,ds = t^2.$$

Continuing, we get

$$\begin{aligned}
\phi_2(t) &= T\phi_1(t) = y_0 + \int_0^t 2s(1+\phi_1(s))\,ds = \int_0^t 2s(1+s^2)\,ds = t^2 + \frac{1}{2}t^4, \\
\phi_3(t) &= T\phi_2(t) = y_0 + \int_0^t 2s(1+\phi_2(s))\,ds = \int_0^t 2s\left(1+s^2+\frac{1}{2}s^4\right)ds = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6, \\
\phi_4(t) &= T\phi_3(t) = y_0 + \int_0^t 2s(1+\phi_3(s))\,ds = \int_0^t 2s\left(1+s^2+\frac{1}{2}s^4+\frac{1}{6}t^6\right)ds = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8.
\end{aligned}$$

Exercise 5. Find the pattern and write out a finite series expression for $\phi_n(t)$. Here one can prove by induction that the pattern you find is the nth iterate function. However, I am more interested in you "seeing" it right now.

Exercise 6. Find a closed form expression for $\lim_{n\to\infty} \phi_n(t)$ and show that it is a solution of the IVP.