110.109 CALCULUS II

Week 9 Lecture Notes: April 2 - April 6

Lecture 1

We start today with an example of the final result from last lecture.

Example 1. *Find the sum of* $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)$*, if it exists.* Here, trying to combine the two terms in the sum will not create an easier fraction to study. However, with the previous result, we can try the following: See if, when we rewrite the series as

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)=\sum_{n=1}^{\infty}\frac{1}{e^n}+\sum_{n=1}^{\infty}\frac{1}{n(n+1)},$$

each of the two separate series on the right converge. If they do, then the original series will converge also, and to the sum of the two individual series on the right. Only if the two on the right converge will this work, but it is worth the effort.

Let's take each of the series on the right separately. First, notice that the series $\sum_{n=1}^{\infty}\frac{1}{e^n}$ looks geometric, due to the fact that the only place we find the n is in the exponent. When we rewrite it:

$$\sum_{n=1}^{\infty}\frac{1}{e^n}=\sum_{n=1}^{\infty}\frac{1}{e}\left(\frac{1}{e}\right)^{n-1},$$

we find that it is indeed geometric, with $a=\frac{1}{e}$, and $r=\frac{1}{e}$. Noting that this choice of r satisfies $|r|<1$, we know that this series converges and the sum is $\frac{a}{1-r}$, so that

$$\sum_{n=1}^{\infty}\frac{1}{e^n}=\sum_{n=1}^{\infty}\frac{1}{e}\left(\frac{1}{e}\right)^{n-1}=\frac{\frac{1}{e}}{1-\frac{1}{e}}=\frac{1}{e-1}.$$

For the second series, try to find a good expression for the partial sum

$$\begin{aligned}s_n=\sum_{i=1}^{n}\frac{1}{i(i+1)}&=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}\\&=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\ldots.\end{aligned}$$

Good luck with that. Outside of this, try to use any structure found in the terms to possibly "uncover" something you can work with. for example, noting that this rational expression of n can be decomposed into two simpler ones via a partial fraction decomposition, we get

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

Date: April 5, 2012.

While this might not look like much help, go back to the partial sum:

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}$$

since all of the intermediate terms cancel out. This is called a *telescoping* sequence, since each succeeding term will kill off all or part of a preceding term. Hence we do have a good expression for the partial sum. So here $s = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, if it exists, where

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1.$$

Hence the second series also converges, and we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right) = \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{e-1} + 1 = \frac{e}{e-1}.$$

So here is a new question: We know that the series $\sum \frac{1}{n}$ diverges, yet $\sum \frac{1}{n(n+1)} = \sum \frac{1}{n^2+n} = \sum \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges to 1.

Question 2. *How about* $\sum_{n=1}^{\infty} \frac{1}{n^2}$*?*

There is no simple expression for the partial sum

$$s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2}$$
$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2}.$$

But notice the following:

The (unbounded) area between the curve of $f(x) = \frac{1}{x^2}$ and the x-axis on the interval $[1, \infty)$ is

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to\infty} 1 - \frac{1}{b} = 1.$$

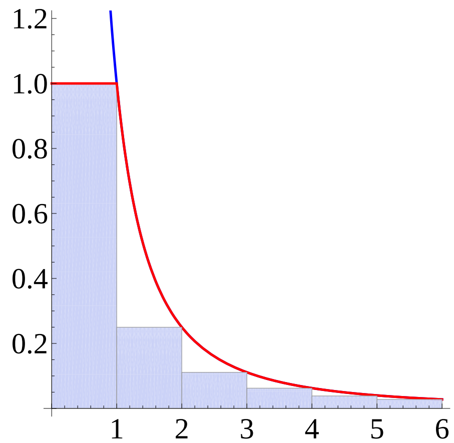
Hence the area of the region between the x-axis and the red curve below in the figure is

$$1 + \int_1^{\infty} \frac{1}{x^2}\, dx = 2.$$

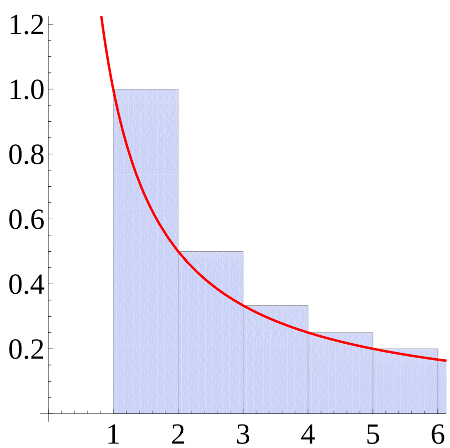
The block shaded region in the figure is a precise, geometric representation of the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The total area of the infinite number of blocks of length 1 and height $\frac{1}{n^2}$, is precisely the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Notice that it is strictly less than the area of the block red region (see all of the gaps?). Hence we must have that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2.$$

This means that the series will converge. Why is this so? First, since the entire series is less than 2, each partial sum will be less than 2. Hence the sequence of partial sums will be bounded by 2. Note that the sequence of partial sums will also be a monotonic sequence (always adding a positive number each time. Hence it must converge as ALL bounded monotonic sequences converge! Thus the series converges. Now we do not know what it will converge to here (actually, we do, and the sum is $\frac{\pi^2}{6}$, but there is no way here to know that here), but just knowing that it does converge is progress. And we found out by comparing to the improper integral of the function which, in some way, generated the sequence.



We can play the same game with the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$. Here is another figure. To study this series, create the function $f(x) = \frac{1}{x}$ so that our series is the sum of all of the terms $a_n = f(n)$ from 1 to ∞. By graphically representing the series as the sum of the areas of the blocks here, we note that



$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\, dx.$$

In this case, the right hand side IS ∞ (the integral diverges). But then this means that the harmonic series will also diverge (the sum will also be infinity) since it will be strictly larger than something that does not converge. Again, we studied the series by instead comparing it to an improper integral. Where the integral went, so went the series. We have the following:

Theorem 3. *Suppose $f(x)$ is a positive, continuous, and decreasing function on $[1, \infty)$, and $a_n = f(n)$. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges iff } \int_1^{\infty} f(x)\, dx \text{ converges.}$$

Some notes:

- This is called the *Integral Test for the Convergence of a Series.*
- This test is great for studying series whose terms look like a function one can integrate.
- This test does not give the sum. While it is tempting to think the improper integral should also give the sum, it does NOT in general.
- Really, where a series starts is not important, and this test works also for series that start later than 1. See the next example.

Example 4. *We now know that* $\sum_{n=2}^{\infty} \frac{1}{n^2}$ *converges, while* $\sum_{n=2}^{\infty} \frac{1}{n}$ *diverges. What about* $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$*?* This one seems to be in between the two. Fortunately, we can use the Integral Test here: Let $f(x) = \frac{1}{x \ln x}$. The improper integral of $f(x)$ is

$$\int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u}\, du.$$

The antiderivative of $\frac{1}{u}$ on the interval $[\ln 2, \infty)$ is $\ln u_+ C$. Hence the antiderivative of $f(x) = \frac{1}{a \ln x}$ on the interval $[2, \infty)$ is $\ln \ln x$. And does this antiderivative have a horizontal asymptote? You should work all of this out, but the answer is no. Hence the integral diverges. Hence also the series diverges.

Question 5. *You know that* $\sum_{n=2}^{\infty} \frac{1}{n^2}$ *converges. What about* $\sum_{n=2}^{\infty} \frac{1}{n^p}$*, when* $p > 2$*?*

Question 6. *Show that* $\sum_{n=2}^{\infty} \frac{1}{e^{n^2}}$ *converges by the integral test. Hint: You cannot do the resulting improper integral directly, but you can use the Comparison test for Improper Integrals to help you. See previous lecture notes.*

Lecture 2: The Comparison Tests

Today we will look at another test for the convergence of a series. This one will seem a lot like what we have done in the past to determine the convergence of sequences as well as improper integrals. The parallels are not coincidental, but part of a grander scheme which is very useful in general mathematical structure: By the careful comparison of a new mathematical object to a known one, one can make fairly dramatic conclusions with relatively simple to use tools.

To start, suppose $\sum b_n$ converges, where the terms $b_n > 0$ for all $n \in \mathbb{N}$, and suppose given another series $\sum a_n$ has terms that satisfy $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.

Question. *Is it enough information to conclude that* $\sum a_n$ *converges?*

To answer this, think about the following: It is true that since the series $\sum b_n$ converges, we know that $\lim_{n\to\infty} b_n = 0$, so that by the Squeezing Theorem for the convergence of sequences, we know also that $\lim_{n\to\infty} a_n = 0$.

Caution: This does NOT ensure the convergence of $\sum a_n$, though, right?

We also know that the partial sums satisfy

$$s_n = b_1 + b_2 + \cdots + b_n > t_n = a_1 + a_2 + \cdots + a_n,$$

since term by term $b_n > a_n$. Using s_n as the partial sum of the series $\sum b_n$, we know that

$$\lim_{n\to\infty} s_n = s,$$

a number (the series converges). But then $\{t_n\}$, the sequence of partial sums of the series $\sum a_n$, is monotonic (each term is positive, so the partial sums are all increasing), and bounded above by s. Hence $\{t_n\}$ must converge as a sequence. But then the series $\sum a_n$ must converge!

In fact, this kind of analysis can work both ways and we get the following:

Theorem 7. *Suppose $\sum a_n$ and $\sum b_n$ are two series with positive terms, and for all n we have $0 \le a_n \le b_n$. Then*

- *If $\sum b_n$ converges, then $\sum a_n$ converges, and*
- *if $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Some notes:

- This is called the *Comparison Test for Series.*
- This test is great for simplifying rational terms. As an example, once can show that $\sum \frac{3}{2n^2+4n+6}$ converges by directly comparing it to the series $\sum \frac{1}{n^2}$. (as an exercise, you should explicitly write out all of the details, and determine why this would work here.)
- This is another way to verify the convergence properties of many p-series. A p-series has the form $\sum \frac{1}{n^p}$ for some choice of real number p. Given any $p > 2$, it is readily seen that
$$\frac{1}{n^p} < \frac{1}{n^2}, \quad n \in \mathbb{N},$$
since the denominator on the right will be smaller than the one on the left, and hence the fraction will always be larger. Of course, we already know which p-series converge by the Integral Test for Convergence of a Series, right?
- The Comparison Test also works for simplifying a complicated expression in the terms of a series by bounding it. For instance, does the series $\sum \frac{\arctan n}{n^4}$ converge? The answer is yes, once one understands that the inverse tangent function is defined

on all of $\mathbb{R}$, and for real numbers greater than 0, takes values in the interval $\left(0, \frac{\pi}{2}\right)$. Hence for all n, we know

$$a_n = \frac{\arctan n}{n^4} < \frac{\frac{\pi}{2}}{n^4} = b_n.$$

And since

$$\sum b_n = \sum \frac{\frac{\pi}{2}}{n^4} = \frac{\pi}{2} \sum \frac{1}{n^4}$$

converges as a p-series with $p > 2$ (see above), we can conclude by the Comparison Test that $\sum \frac{\arctan n}{n^4}$ also converges.

Example 8. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ converge?

To understand this, let $a_n = \frac{1}{n!}$. We compare this series to $\sum b_n = \sum \frac{1}{n^2}$. Here

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \cdots$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1 \cdot 1} + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \frac{1}{5 \cdot 5} + \frac{1}{6 \cdot 6} + \cdots .$$

After the third term, once can see that $a_n < b_n$ for all $n > 3$, and all terms of each series are positive. Since $\sum b_n$ converges, we have by the Comparison Test for Series that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

So what does it converge to? Well, this test does not tell us. But by the end of the semester, you WILL know that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1.$$

Now, try to compare these two series that have these terms:

$$a_n = \frac{1}{n^2 - 1}, \quad b_n = \frac{1}{n^2}.$$

We know that $\sum b_n$ converges as a series. However, we cannot use the Comparison Test directly since for all $n > 1$,

$$a_n = \frac{1}{n^2 - 1} > \frac{1}{n^2} = b_n.$$

Comparing a convergent series to one known to be larger then it does not work. If you were to use your intuition, however, you might conclude that $\sum a_n$ should probably converge, being as it is very much like $\sum b_n$; The extra 1 in the denominator should not make much of a difference when n gets quite large...). In a way, this is because while for any convergent series, the terms must "go to 0", it is the rate at which these terms go to 0 which determine whether the series converges or not. For example, geometric series like $\sum \left(\frac{1}{2}\right)^n$ converge because the terms collapse to 0 quickly. Contrast this with the non-convergent Harmonic Series $\sum \frac{1}{n}$, where the terms go to 0 much more slowly. Measuring the relative rates at which two series, one you know whether it converges or not, the other you want to know, may help you determine whether a new series converges or not. And how does one measure the rate at which two sequences "go to 0"? BY studying their ratios, term by term. Where does that sequence go? This is the essence of the *Limit Comparison Test*, given by the next theorem:

Theorem 9. *Suppose $\sum a_n$, $\sum b_n$ are two series with positive terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0,$$

then either both series converge or both diverge.

Let's prove this Limit Comparison Test before discussing its ramifications:

Proof. Let m and M be two positive constants, chosen so that $m < c < M$. If c is a non-zero number, we can always find a large one and a smaller one which bound it. It doesn't really matter which choice one makes. Given these two choices, it must them be the case that there will be some integer N, where for all $n > N$, we will have

$$m < \frac{a_n}{b_n} < M.$$

Why is this the case? Think about what it means for the sequence $\left\{\frac{a_n}{b_n}\right\}$ to converge. Go back to the convergence criteria for a sequence if you need to. Given this value of N then, we will also have that for $n > N$,

$$mb_n < a_n < Mb_n.$$

Here is where it gets interesting then.

Suppose $\sum b_n$ converges. Then the new series $\sum Mb_n = M \sum b_n$ also converges. But then by the Comparison Test, since $a_n < Mb_n$, we will have that $\sum a_n$ converges.

No Suppose that $\sum b_n$ diverges. Then the new series $\sum mb_n = m \sum b_n$ also diverges. But then by the Comparison Test, since $mb_n < a_n$, we will have that $\sum a_n$ diverges. □

Example 10. Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ converge?

(Special note: Do you know why I started the series at $n = 2$ and not $n = 1$?) To answer this, recall that $\sum \frac{1}{n^2}$ converges. Denote, like above, $a_n = \frac{1}{n^2-1}$ and $b_n = \frac{1}{n^2}$. Then, since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1 \neq 0,$$

it follows by the Limit Comparison Test that $\sum a_n$ also converges.

Some Notes:

- The value of c is really meaningless here, as long as it is a non-zero positive number. In fact, if one were to make the "other" choices for a_n and b_n, then the limit of the ratios would lead to a sequence of reciprocals. The limit would then be the reciprocal of the original number c, which would still be a non-zero, positive number.
- For terms that look like rational functionals, you can determine convergence almost by sight: Consider the new series whose terms are rational function using only the highest degree terms in the numerator and denominator. And use the Limit Comparison Test with this new series and the original.

Example 11. Let $a_n = \frac{6n^3+1}{12n^4+7n^2}$. Does $\sum a_n$ converge?

For the limit Comparison Test, choose $b_n = \frac{6n^3}{12n^4} = \frac{1}{2n}$. Just drop all lower degree terms to create a new set of series terms and simplify. There is no need to worry about which is larger, term by term. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{6n^3+1}{12n^4+7n^2}}{\frac{1}{2n}} = \lim_{n\to\infty} \frac{6n^3+1}{12n^4+7n^2} \cdot \frac{2n}{1} = \lim_{n\to\infty} \frac{12n^4+2n}{12n^4+7n^2} = 1 \neq 0.$$

Hence $\sum a_n$ converges iff $\sum b_n$ converges. And since we know

$$\sum b_n = \sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$$

diverges (it is the Harmonic Series), we now know that $\sum a_n$ also diverges.

Next time, we will look at a special kind of non-positive series.

Lecture 3: Alternating Series