110.109 CALCULUS II

Week 8 Lecture Notes: March 26 - March 30

We begin this week by continuing where we left off before the break: Using the Squeezing Theorem to determine the convergence of a sequence. Another example:

Example 1. *Does* $\{c_n\} = \left\{\frac{\ln n}{(n+2)^2}\right\}$ *converge?* This sequence is not as clear as the last example, as both the numerator and the denominator get large as we progress along the sequence. But we can still compare this sequence to others in a straightforward manner.

Notice that for $n > 1$, $c_n > 0$. And notice that, again for $n > 1$, in general

$$\frac{1}{(n+2)^2} < \frac{1}{n^2}.$$

Thus for $n > 1$,

$$0 < c_n = \frac{\ln n}{(n+2)^2} < \frac{\ln n}{n^2}.$$

So let $a_n = 0$ and $b_n = \frac{\ln n}{n^2}$. It is obvious that $\{a_n\} \longrightarrow 0$. It is less obvious that $\{b_n\} \longrightarrow 0$. I leave this latter calculation to you, but note that the function $f(x) = \frac{\ln x}{x^2}$ should be very helpful to you. Then by squeezing, we also know that $\{c_n\} \longrightarrow 0$.

Stripping off alternating terms. While many of our examples so far centered on sequence rules that allowed us to use a corresponding function to study their possible convergence, many sequences are not easily converted into functions. For example, sequences with alternating signs, like our original $b_n = \frac{(-1)^n}{2^n}$ do not work so well as continuous functions by replacing n with x. However, there is hope, and it involves stripping away the alternating term.

Theorem 2. *If* $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

One must be careful here to apply this theorem correctly. It ONLY works when the limit of the absolute values of the sequence is 0, and not another number. The quintessential example of this is the sequence $a_n = (-1)^n$. Here

$$\lim_{n\to\infty} |(-1)^n| = \lim_{n\to\infty} 1 = 1, \text{ but } \lim_{n\to\infty} (-1)^n \text{ does not exist!}$$

Example 3. *Show* $b_n = \frac{(-1)^n}{2^n}$ *converges to 0.* We use the theorem directly, since

$$\lim_{n\to\infty} |b_n| = \lim_{n\to\infty} \left|\frac{(-1)^n}{2^n}\right| = \lim_{n\to\infty} \frac{1}{2^n} = 0 \text{ (Why?)}$$

hence by the theorem, so does b_n.

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Passing a limit though a continuous function. One of the endearing properties of limits is that one can pass a limit through a continuous function. Essentially this is Theorem 8 in Section 2.5 on continuity in the text (see page 125). For sequences, this remains true:

Theorem 4. *If* $\lim_{n\to\infty} a_n = L$ *and* f *is a function continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

This is particularly useful when the sequence rule is quite complicated and consists of a composition of functions.

Example 5. *Show* $a_n = \tan\left(\frac{\pi n^2}{9n+4n^2-12}\right)$ *converges and find the limit.* We will use this theorem to solve this problem by first looking at the sequence $b_n = \frac{\pi n^2}{9n+4n^2-12}$. We find that this sequence is convergent and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\pi n^2}{4n^2 + 9n - 12} = \frac{\pi}{4} \text{ (again, why?)}.$$

And since $f(x) = \tan x$ is continuous at $x = \frac{\pi}{4}$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan(b_n) = \tan\left(\lim_{n\to\infty} b_n\right) = \tan\frac{\pi}{4} = 1.$$

To go farther, let's look more closely at some of the properties of certain sequences.

Definition 6. A sequence $\{a_n\}$ is called *increasing* if, for all $n \geq 1$, we have $a_n < a_{n+1}$. $\{a_n\}$ is called *decreasing* if, for all $n \geq 1$, we have $a_n > a_{n+1}$. If either is the case, the sequence is called *monotonic*.

Definition 7. A sequence $\{a_n\}$ is said to be *bounded above* if there is a constant M, where $a_n \leq M$ for all $n \geq 1$. $\{a_n\}$ is said to be *bounded below* if there is a constant m, where $a_n \geq m$ for all $n \geq 1$. If both are true, the sequence is simply called *bounded*.

This immediately leads to yet another theorem on the convergence of sequences:

Theorem 8. *Every bounded monotonic sequence converges.*

Really, think of this as follows: If a sequence must always rise, and yet never cross a particular value, then it must converge. Think of a helium balloon in a closed room. It cannot go through the ceiling, and yet will constantly rise (at least until the helium leaks out, that is). Hence at least in the limit, it will have to reach a steady height.

Sequences can also be defined recursively, as in the value of a particular term is a function, not of the index, but of the value of the previous term or some combination of previous terms of the sequence. For example, Example 13 in the section defines the sequence

$$a_{n+1} = \frac{1}{2}(a_n + 6), \quad a_1 = 2.$$

And Example 3 in the book defines the Fibonacci Sequence as

$$f_{n+2} = f_{n+1} + f_n, \quad f_0 = 1, f_1 = 1.$$

Notice that the rule in both of these cases specifies the value of a term as a function of the previous terms. Notice also that for this to work, we would always need to “start” somewhere. The obvious advantage to this is that for different starting values, we can get many different sequences, whose properties like convergence may depend on where one starts. The obvious disadvantage to this is that to know the value of a term whose index is large would require knowing ALL of the previous terms' values. That could be problematic. Compare that to the functional form of most of our sequences that we have defined so far.

Pay attention to Example 13 in the text. There are great conceptual ideas in this discussion. One such idea involves the notion of *mathematical induction*: a form of rigorous deductive reasoning that states that a statement is deemed true for all of the natural numbers if it is true for the smallest of them, and if it is also true at any stage, then it is true also at the next stage. It is a great way to establish a general result with only a couple of calculations, and forms one of the standard ways that mathematicians prove things.

In the case at hand, let's study the sequence of Example 13. Notice first what the sequence will be if we start with $a_1 = 6$. We get

$$a_{n+1} = \frac{1}{2}(a_n + 6), \quad a_1 = 6 \quad \Longrightarrow \quad a_n = 6, \text{ for all } n \in \mathbb{N}.$$

Now suppose that we start with some positive starting value for the sequence $0 \leq a_1 < 6$. Then, it is the case that

$$0 \leq a_2 = \frac{1}{2}(a_1 + 6) < \frac{1}{2}(6 + 6) = 6.$$

In fact, let's suppose that somewhere along our sequence the nth term satisfies $0 \leq a_n < 6$. Then $6 \leq a_n + 6 < 6 + 6 = 12$. Then also $\frac{1}{2}(6) = 3 \leq \frac{1}{2}(a_n + 6) < \frac{1}{2}(12) = 6$. But the middle part of the inequality IS a_{n+1}. Hence we have shown that whenever a term in the sequence is between 0 and 6, then so is its successor. Hence if we start in the interval $[0, 6]$, we never leave it. By induction, the sequence $\{a_n\}$ is **bounded**.

Let's play this game again in a different way. Notice that for our original sequence starting at $a_1 = 2$, we get $a_2 = \frac{1}{2}(2 + 6) = 4$, which is greater than 2. we can duplicate this again through out the sequence to see how the sequence progresses.

Suppose we knew that somewhere along the sequence we had $a_{n-1} < a_n$. Then we can say

$$\begin{aligned} a_{n-1} &< a_n \\ a_{n-1} + 6 &< a_n + 6 \\ \frac{1}{2}(a_{n-1} + 6) &< \frac{1}{2}(a_n + 6) \\ a_n &< a_{n+1}. \end{aligned}$$

Thus if somewhere along the way we know the sequence rises, then at the next stage the sequence continues to rise. Since it started by rising, it will be forced to rise forever (but never get larger than 6, right?). Hence, by induction, the sequence is increasing, and **monotonic**.

Put these together and you get by the previous theorem that this sequence must converge. It turns out that it converges to 6, actually.

Lecture 2: Section 10.2 Series

Today we start the discussion on a series, basically the sum of the terms of a sequence:

Definition 9. The sum of all of the terms in a sequence $\{a_n\}$ is called an (infinite) series, and denoted

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \qquad \text{or simply} \qquad \sum a_n.$$

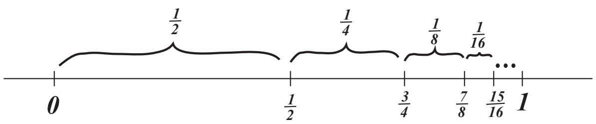
Note that sometimes this sum doesn't exist:

$$a_n = n, \qquad \text{so that} \qquad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots = \infty,$$

and sometimes it does:

$$a_n = \frac{1}{2^n}. \qquad \text{Here} \qquad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1??$$

To see geometrically this "sum", look on the number line. Adding $\frac{1}{2}$ to 0 and then successively adding half of the previous amount in turn, notice how you will never exceed 1. Yet if you believe that you will finish somewhere before 1, convince yourself that you will eventually pass that number. This looks like a limit-type process and you are correct. The sum is indeed 1, but to see this will take a bit of work.



First, denote by s_n the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$, so that

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

As we push n toward ∞, these partial sums will tend toward a total sum, at least when the sum actually exists.

Example 10. For $a_n = \frac{1}{2^n}$, we have

$$s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \quad s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, \quad \cdots \quad .$$

Do you sense a pattern? We have

$$s_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

For a given series, the set of all partial sums forms a new sequence $\{s_n\}_{n=1}^{\infty}$. We can say that where this sequence goes, so will go the series.

Definition 11. Given a series $\sum_{n=1}^{\infty} a_n$ with its associated sequence of partial sums $\{s_n\}$, if $\lim_{n\to\infty} s_n = s$ exists (so that $\{s_n\}$ is convergent), then the series is convergent and

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = s.$$

We call s the sum of the series, if it exists. If the sum does not exist, we say that the series is divergent.

Example 12. Again, back to $a_n = \frac{1}{2^n}$. we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n},$$

as long as the two limits on the right hand side exist. They do, and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 - 0 = 1.$$

The series is convergent and its sum is 1.

Definition 13. A series of the form

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \quad a \neq 0,$$

is called a *geometric* series.

In a geometric series, each term is the product of the previous term and the factor r, so that the terms of the series look like $a_{n+1} = ra_n$ for all $n \in \mathbb{N}$. As an example, the previous series we have been discussing, namely

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

is geometric with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.

A huge question involving a series in general, and a geometric series in particular, is: Does the series converge? To answer this, we need to appeal to the sequence of partial sums. Let's look specifically at how the value of r determines whether the geometric series converges or not.

Case 1. *Let* $r = 1$. Then for $a \neq 0$ (there is not much to discuss in the case that $a = 0$, no?) the series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} a$. The partial sum is then

$$s_n = a + a + \cdots + a = na,$$

and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} na = a \lim_{n\to\infty} n = \infty.$$

Hence the series diverges.

Case 2. *Let* $r \neq 1$. Here then the series has its partial sum

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

This is not so easy to manage, but there are clever ways to play with this expression. Here is one; note that

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Then

$$s_n - rs_n = a - ar^n,$$

since all of the other terms cancl out (this is why the difference here is clever!). Now, solve for s_n to get $s_n(1-r) = a(1-r^n)$, or

$$s_n = \frac{a(1-r^n)}{1-r}.$$

So does the series with this partial sum sequence converge? we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \lim_{n\to\infty} 1 - r^n,$$

which will converge precisely when $|r| < 1$, and diverge precisely when $|r| \geq 1$.

Note: As a special case, let $r = -1$. Then $\lim_{n\to\infty} 1 - r^n$ does not converge, but not because the sequence of partial sums gets large. Instead, here,

$$s_n = \frac{a(1 - r^n)}{1 - r} = \begin{cases} a & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

The sequence of partial sums does not exist here, even though the sequence is bounded. Hence it is still the case that the geometric series diverges for $r = -1$.

Here are some examples of geometric series that you have already played with.

Example 14. *Exponential Functions.* Let $f(x) = b^x$, where $b > 0$. Then we can create a sequence $a_n = f(n) = b^n$. This sequence is geometric, with the initial constant $a = b$, and the base $r = b$, and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b^n = \sum_{n=1}^{\infty} b\,(b)^{n-1}.$$

Again, this series will converge iff $b < 1$. And it will converge precisely to $\frac{a}{1-r} = \frac{b}{1-b}$. Thus, there is a close relationship between exponential functions and geometric series. we say that exponential function are examples of *geometric growth*.

Exercise 1. Find a value for b so that the series $\sum_{n=1}^{\infty} b^n = 5$.

Solution. While this may seem straightforward, there is one thing to consider. To use the sum formula for a geometric series, it is absolutely necessary to correctly identify the values for a as well as r: Here again

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} b\,(b)^{n-1},$$

so that $a = r = b$. Now we can use the folmula

$$\sum_{n=1}^{\infty} b\,(b)^{n-1} = \frac{b}{1-b} = 5.$$

The last equation is easy to solve with a little algebra, and we find $b = \frac{5}{6}$ (check this!).

Example 15. *Decimal representations are geometric series.* The rational number $\frac{5}{9}$ has the decimal representation $.55555\overline{5}$, where the bar over the last 5 indicates that the pattern continues. But recall the definition of a decimal representation means that

$$.55555\overline{5} = \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \cdots.$$

This is simply a geometric sequence with $a = \frac{5}{10}$ and $r = \frac{1}{10}$, and

$$.55555\overline{5} = \sum_{n=1}^{\infty} \frac{5}{10}\left(\frac{1}{10}\right)^{n-1}.$$

Notice again that this geometric sequence converges since $|r| = \left|\frac{1}{10}\right| < 1$. And it converges to the quantity $\frac{a}{1-r}$, which in this case is

$$\sum_{n=1}^{\infty} \frac{5}{10}\left(\frac{1}{10}\right)^{n-1} = \frac{a}{1-r} = \frac{\frac{5}{10}}{1-\frac{1}{10}} = \frac{\frac{5}{10}}{\frac{9}{10}} = \frac{5}{9},$$

as it should.

Exercise 2. Determine whether the series $\sum_{n=1}^{\infty} 5(2^{n+2})(3^{-n+1})$ converges. If it does, then find the sum.

Think about this. We will do it next time.

Lecture 3: Section 10.2 Series (cont'd.)

Example 16. *Determine whether the series* $\sum_{n=1}^{\infty} 5(2^{n+2})(3^{-n+1})$ *converges. If it does, then find the sum.* Due to the presence of the index variable n in the exponent, and not anywhere else, this series has a good chance of being a geometric series, and if we can put it into the form of a geometric series (by correctly identifying both a and r), then perhaps we can answer the question. To this end, let's try to manipulate the term $a_n = 5(2^{n+2})(3^{-n+1})$. Here we have

$$a_n = 5(2^{n+2})(3^{-n+1}) = 5 \cdot 2^3 \cdot 2^{n-1} \cdot \frac{1}{3^{n-1}} = 40 \cdot \left(\frac{2}{3}\right)^{n-1}.$$

Notice that we have stripped out a 2^3 from 2^{n+2}, knowing that $2^3 2^{n-1} = 2^{3+(n-1)} = 2^{n+2}$, and $3^{-n+1} = 3^{-(n-1)} = \left(\frac{1}{3}\right)^{n-1}$. Putting these together, we wind up with a term that meets the form of a geometric series, with $a = 40$ and $r = \frac{2}{3}$. This series converges since $|r| < 1$, and the sum is

$$\sum_{n=1}^{\infty} 5(2^{n+2})(3^{-n+1}) = \sum_{n=1}^{\infty} 40\left(\frac{2}{3}\right)^{n-1} = \frac{40}{1-\frac{2}{3}} = 120.$$

Now if the series is NOT geometric, then we cannot use the convergence criterion used above. We can still appeal to the original notion that if we can find a good expression for the terms of the sequence of partial sums s_n, we can try to determine if the sequence has a limit. If it does, then the series also converges and to the limit of the sequence of partial sums.

Outside of this, sometimes one needs to simply be clever and look for patterns and structure to exploit to see if a given series may converge or not. Examples 6 and 7 in the text are two good examples of this type of cleverness. Example 6 deals with a series called a telescoping series, where each succeeding pair of terms of the series almost cancel each other out, leaving very little but a couple of terms in the partial sum. This gives a useful expression for the partial sum and the convergence of the series is determined by analyzing the sequence of partial sums.

In example 7, they study the Harmonic Series. Let's do that one in detail:

Example 17. *The Harmonic Series.* We saw that the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converged, even though it was a sum of an infinite number of positive numbers. The point was that the positive numbers were decreasing quickly and quickly enough that the ultimate sums didn't exceed (in this case) 1. Here, we introduce the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

Here, the terms of the sequence are all still positive numbers which form a decreasing sequence, but they decrease much more slowly. Also, there is no easy way to write out a useable expression for the partial sum

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

And the sequence is certainly not geometric.

It turns out that the series diverges, and in this case, that means that the sum is infinite. How to show this involves showing that the sequence of partial sums eventually exceeds every integer. Indeed, we will do this by limiting ourselves to only certain of the partial sums, and noting a useful pattern.

To start, notice that the first two partial sums have a pattern to them, if seen in a certain way:

$$s_1 = 1 = 1 + \frac{0}{2}, \qquad \text{and} \qquad s_2 = 1 + \frac{1}{2}.$$

I like the pattern, and notice that if we skip to s_4, we get

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2} = 2$$

Really, this says nothing more than the fact that $s_4 > 2$.

But there is a pattern here: If we pass to s_8, we notice that

$$s_8 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > 1+\frac{1}{2}+\left(\frac{1}{4} + \frac{1}{4}\right)+\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1+\frac{3}{2}$$

which means that $s_8 > 2.5$. We can continue this pattern by next looking at s_{16} and noting that it is strictly larger than $1 + \frac{4}{2} = 3$, then to s_{32}, noting that it is strictly larger than

$1 + \frac{5}{2}$, and generalizing to the pattern that at the partial sum of the 2^nth term, we have

$$s_{2^n} > 1 + \frac{n}{2}.$$

Do you see how this is helpful. Say I wanted to know when the partial sum of the harmonic series passes the sum of 100. Then I simply solve $100 = 1 + \frac{n}{2}$, This give me $n = 198$. So then I know that the partial sum

$$s_{2^{198}} > 100.$$

Since I can do this for every number n, eventually, my partial sum of the harmonic series will pass every integer. This means that $\lim_{n\to\infty} s_n = \infty$, so that the harmonic series *diverges.*

By the way, do not worry about having to be so clever in finding ways to show that a series converges or not. In time, you will develop such abilities. For now, know that there are many useful tools to determine whether a series converges or not. The rest of this section and the next few will develop some of those ways. Here is one:

Theorem 18. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

All this theorem really says is that the only way that a series can converge is if the sequence of partial sums has a limit. That sequence of partial sums can only have a limit if the sequence tends to a number . That means that the things we add to each partial sum to get the next one need to decay away to 0 as the index n goes to infinity. Makes sense, but be very careful here:

CAUTION. The converse of this statement is NOT true!

The statement: "If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges" is false! The harmonic series is one example where the terms go to 0, but the series diverges. Be very careful here.

Instead, the contrapositive of the theorem is true. The contrapositive of a conditional statement is formed by negating both antecedent (the "if part) and the consequent (and "and" part), and switching them. We get:

Theorem 19. *If* $\lim_{n\to\infty} a_n \neq 0$ *(or does not exist), then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

One can call this The Divergence Test for a series. It is very effective.

Example 20. *Show* $\sum_{n=1}^{\infty} \frac{1+n^2}{5n^2+6n}$ *diverges.* Here, we see

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{5n^2+6n} = \frac{1}{5} \neq 0.$$

At this point, we know by the Divergence Test that the series will diverge.

Other conclusions to mention? Well, much like limits, series behave well in sums, differences, and the like (after all, there ARE defined by limits of their partial sums).

Suppose $\sum a_n$ and $\sum b_n$ are two convergent series (this only works when these sums are convergent). Then

(1) $\sum ca_n = c \sum a_n$ converges, and
(2) $\sum a_n \pm b_n = \sum a_n \pm \sum b_n$ also converges.

Here, the symbol $\pm$ means that when there is a "plus" on the left-hand side, there is also a plus on the right, and when there is a "minus" on the left, there is also one on the right. This should all make sense, but again, only if the individual sums exist. There is no need to prove this result. Just keep it in mind.

Next time, we will do another example.