110.109 CALCULUS II

Week 7 Lecture Notes: March 12 - March 16

Improper Integrals (cont'd.)

Going back to the definition of an *improper integral*, we can add a second part:

Definition 1. For $f(x)$ continuous for all $x \geq a$, the improper integral

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

provided the limit exists. If the limit exists, we say the improper integral *converges*. Else, we say it *diverges*.

For $f(x)$ continuous for all $x \leq b$, the improper integral

$$\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty} \int_a^b f(x)\,dx,$$

provided the limit exists.

Example 2. Calculate $\displaystyle\int_0^\infty te^{-t}\,dt$, if it exists (Note that this is much like Example 7.8.2 in the book, but going the other way). Here the function is continuous everywhere, but the integral is improper. Hence we need to appeal to the limit:

$$\int_0^\infty te^{-t}\,dt = \lim_{b\to\infty}\int_0^b te^{-t}\,dt.$$

Using the technique of Integration by Parts, where $f(t) = t$, and $g'(t) = e^{-t}$, we get $f'(t) = 1$, and $g(t) = -e^{-t}$, and

$$\begin{aligned}
\lim_{b\to\infty}\int_0^b te^{-t}\,dt &= \lim_{b\to\infty}\left(-te^{-t}\Big|_0^b - \int_0^b -e^{-t}\,dt\right) \\
&= \lim_{b\to\infty}\left(-te^{-t} - e^{-t}\,dt\right)\Big|_0^b \\
&= \lim_{b\to\infty}\left(\left(-be^{-b} - e^{-b}\right) - (0-1)\right) \\
&= \lim_{b\to\infty}\left(-\frac{b+1}{e^b} + 1\right) = 1.
\end{aligned}$$

How would you show the last step? Try L'Hospital's Rule. Hence, this integral converges and its value is 1.

Date: March 20, 2012.

Example 3. Calculate $\displaystyle\int_{-\infty}^{0} \sin x\, dx$, if it exists. Here the function is continuous everywhere, and the integral is again improper. Appealing to the limit:

$$\int_{-\infty}^{0} \sin x\, dx = \lim_{a\to-\infty} \int_{a}^{0} \sin x\, dx = \lim_{a\to-\infty} \left(-\cos x\right)\Big|_{a}^{0} = \lim_{a\to-\infty} \left(-\cos 0 + \cos a\right).$$

But this limit does not exist for a different reason; the cosine function does not have a limit at infinity (it does not have a horizontal asymptote!). Hence the improper integral diverges.

Exercise 1. Show, for $p \in \mathbb{R}$, the improper integral $\displaystyle\int_{1}^{\infty} \frac{1}{x^p}\, dx$ converges if $p > 1$ and diverges is $p \leq 1$.

And finally, an integral is also improper if both of its limits "run off the page". Well, to handle this case, remember that we can always break up an interval given by the limits into two pieces, evaluate the definite integral on both pieces, and then add the results. This is important for the following definition.

Definition 4. Suppose $f(x)$ is continuous on $\mathbb{R}$. Then for ANY choice of $c \in \mathbb{R}$, the improper integral

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_{c}^{\infty} f(x)\, dx,$$

provided that BOTH of the improper integrals on the right-hand-side exist.

Example 5. Calculate $\displaystyle\int_{-\infty}^{\infty} 3xe^{-x^2}\, dx$, if is exists. Here, the integrand is continuous on the entire real line (so the integral will exist on ANY finite interval), and the integral is improper (both limits need to be addressed). Choose our intermediate value to be 0 for symmetry (though this does not matter), and

$$\begin{aligned}\int_{-\infty}^{\infty} 3xe^{-x^2}\, dx &= \int_{-\infty}^{0} 3xe^{-x^2}\, dx + \int_{0}^{\infty} 3xe^{-x^2}\, dx \\ &= \lim_{a\to-\infty} \int_{a}^{0} 3xe^{-x^2}\, dx + \lim_{b\to\infty} \int_{0}^{b} 3xe^{-x^2}\, dx.\end{aligned}$$

Here, a straightforward substitution of $u = x^2$, $du = 2x\, dx$ is very helpful. Working just with the antiderivative for a minute, we get

$$\int 3xe^{-x^2}\, dx = \int \frac{3}{2} e^{-u}\, du = -\frac{3}{2} e^{-u} + C = -\frac{3}{2} e^{-x^2} + C.$$

Back to our improper integral, we have

$$\begin{aligned}\int_{-\infty}^{\infty} 3xe^{-x^2}\,dx &= \lim_{a\to-\infty}\int_a^0 3xe^{-x^2}\,dx + \lim_{b\to\infty}\int_0^b 3xe^{-x^2}\,dx \\ &= \lim_{a\to-\infty}\left(\left(-\frac{3}{2}e^{-x^2}\right)\Bigg|_a^0\right) + \lim_{b\to\infty}\left(\left(-\frac{3}{2}e^{-x^2}\right)\Bigg|_0^b\right) \\ &= \lim_{a\to-\infty}\left(-\frac{3}{2}+\frac{3}{2}e^{-a^2}\right) + \lim_{b\to\infty}\left(-\frac{3}{2}e^{-b^2}+\frac{3}{2}\right) = \frac{3}{2}+\frac{3}{2} = 3.\end{aligned}$$

The improper integral converges to 3.

Next, I defined another way for an integral to be improper. Go back to the example of an integral where the integrands is $f(x) = \frac{1}{x^2}$, but this time choose to integrate on the interval $[b, 1]$, where $0 < b \le 1$. For a choice of b here, we get

$$\int_b^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_b^1 = -1 + \frac{1}{b} > 0.$$

We know it is greater than 0 since the entire graph of $f(x) > 0$, and also for the last expression $\frac{1}{b} > 1$ (why?)

Question 6. *What if we pushed b all the way back to 0?*

Again, we would produce a problem with the limits of the integral. And again, the problem would involve an interpretation of the definite integral as the area of an unbounded region. However, this time, the region is unbounded in the vertical direction due to the vertical asymptote at $x = 0$. To deal with this situation, we employ the same trick as before, noting that the definite integral is perfectly well-defined for all positive values of b: define

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to0^+}\int_b^1 \frac{1}{x^2}\,dx = \lim_{b\to0^+}\left(-\frac{1}{x}\Big|_b^1\right) = \lim_{b\to0^+}\left(-1+\frac{1}{b}\right) = \lim_{b\to0^+}\left(\frac{1-b}{b}\right).$$

Finishing this calculation means evaluating the limit at the end of the equations above. We will get ∞, and can interpret this as the area between $f(x)$ and the x-axis grows without bound as we push the lower limit back to 0. This is an example of another type of improper integral, where the integrand is not defined at one or both of the limits.

Example 7. *Do the same thing by replacing the integrand with the new function $g(x) = \frac{1}{\sqrt[3]{x}}$.* Note that the graph look remarkably similar, and there is again a vertical asymptote of $g(x)$ at $x = 0$. However, this time, we get

$$\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{b\to0^+}\int_b^1 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{b\to0^+}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_b^1\right) = \lim_{b\to0^+}\left(\frac{3}{2}-\frac{3}{2}\sqrt[3]{b^2}\right) = \frac{3}{2}.$$

Here the resulting limit does exist (the function $\sqrt[3]{x^2}$ is continuous from the right at $x = 0$). The interpretation is that the area of the unbounded region between the curve and the x-axis is finite and is $\frac{3}{2}$.

Exercise 2. Show, for $p \in \mathbb{R}$, the improper integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges if $p < 1$ and diverges is $p \geq 1$.

Exercise 3. Show, for ANY $p \in \mathbb{R}$, the improper integral $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$ diverges.

Example 8. *How about* $h(x) = \frac{(x+1)(x-2)}{x-2}$ *on the interval* $[1,3]$*?* Here both limits of the integral of $h(x)$ would be fine. However, there is a point *inside* the interval (namely $x = 2$) where the function is not defined. We cannot simply ignore this point. We can, however, adjust the calculation to accommodate it knowing the Sum Law for Integrals (Section 5.2, page 374). We can write

$$\int_1^3 \frac{(x+1)(x-2)}{x-2}\,dx = \int_1^2 \frac{(x+1)(x-2)}{x-2}\,dx + \int_2^3 \frac{(x+1)(x-2)}{x-2}\,dx,$$

knowing that both of the integrals on the right-hand side are improper. Keep in mind that they are improper because one of their limits is outside of the domain of the function, and not because there is an asymptote there (there isn't in this case). In this calculation, we get

$$\begin{aligned}\int_1^3 \frac{(x+1)(x-2)}{x-2}\,dx &= \int_1^2 \frac{(x+1)(x-2)}{x-2}\,dx + \int_2^3 \frac{(x+1)(x-2)}{x-2}\,dx \\ &= \lim_{c\to 2^-}\int_1^c \frac{(x+1)(x-2)}{x-2}\,dx + \lim_{c\to 2^+}\int_c^3 \frac{(x+1)(x-2)}{x-2}\,dx \\ &= \lim_{c\to 2^-}\int_1^c (x+1)\,dx + \lim_{c\to 2^+}\int_c^3 (x+1)\,dx.\end{aligned}$$

Notice here two things: (1) We set up the limits only from one side. In each case, we only need to evaluate the limit from the side *within* the interval of integration. This is very important. And (2), since we are now off the "bad" point at $x = 2$, the function $h(x)$ is equivalent to $x + 1$, and we can simplify the integrand. Continuing the calculation

$$\begin{aligned}\int_1^3 \frac{(x+1)(x-2)}{x-2}\,dx &= \lim_{c\to 2^-}\int_1^c (x+1)\,dx + \lim_{c\to 2^+}\int_c^3 (x+1)\,dx \\ &= \lim_{c\to 2^-}\left(\frac{x^2}{2} + x\Big|_1^c\right) + \lim_{c\to 2^+}\left(\frac{x^2}{2} + x\Big|_c^3\right) \\ &= \lim_{c\to 2^-}\left(\frac{c^2+2c}{2} - \frac{3}{2}\right) + \lim_{c\to 2^+}\left(\frac{15}{2} - \frac{c^2+2c}{2}\right).\end{aligned}$$

Both of these limits exist as the expressions are perfectly continuous at $x = 2$. To finish,

$$\int_1^3 \frac{(x+1)(x-2)}{x-2}\,dx = \lim_{c\to 2^-}\left(\frac{c^2+2c}{2} - \frac{3}{2}\right) + \lim_{c\to 2^+}\left(\frac{15}{2} - \frac{c^2+2c}{2}\right)$$
$$= \left(\frac{8}{2} - \frac{3}{2}\right) + \left(\frac{15}{2} - \frac{8}{2}\right) = \frac{12}{2} = 6.$$

Hence the improper integral exists and its value is 6.

So let's now define exactly what I mean by an improper (definite) integral of this kind, and the plan for evaluating it, if possible.

Definition 9. If $f(x)$ is continuous on $[a, b)$ and discontinuous at $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx,$$

provided the limit exists.

Definition 10. If $f(x)$ is continuous on $(a, b]$ and discontinuous at $x = a$, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx,$$

provided the limit exists.

Note that, like before, if the limit exists in either case, we say that the improper integral *converges*. If the limit does not exist, then the integral *diverges*.

Definition 11. If $f(x)$ is continuous on $[a, b]$ except at the point $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where at least one of the integrals on the right-hand side are improper.

We will continue next time.

Lecture 2: Improper Integrals (cont'd) and Sequences

Example 12. *Calculate* $\displaystyle\int_0^{2\pi} f(x)\,dx$*, if possible, for the piecewise-defined function*

$$f(x) = \begin{cases} x & \text{if } \; x \in (-\infty, \pi) \\ \cos(x-\pi) & \text{if } \; x \in [\pi, \infty) \end{cases}.$$

Here we not that the function is discontinuous at $x = \pi$, and use the last definition to write

$$\begin{aligned}\int_0^{2\pi} f(x)\,dx &= \int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} \cos(x-\pi)\,dx \\ &= \lim_{c\to\pi^-} \int_0^{c} x\,dx + \int_{\pi}^{2\pi} \cos(x-\pi)\,dx\end{aligned}$$

since only one of the integrals is actually improper ($f(x)$ is continuous from the right at $x = \pi$. Hence the last integral on the right is okay). Continuing, we get

$$\begin{aligned}\int_0^{2\pi} f(x)\,dx &= \lim_{c\to\pi^-} \int_0^{c} x\,dx + \int_{\pi}^{2\pi} \cos(x-\pi)\,dx \\ &= \lim_{c\to\pi^-} \left(\frac{x^2}{2}\bigg|_0^c\right) + \left(\sin(x-\pi)\Big|_{\pi}^{2\pi}\right) \\ &= \lim_{c\to\pi^-} \left(\frac{c^2}{2}\right) = \frac{\pi^2}{2}.\end{aligned}$$

Remark 13. You may have noticed (like in Examples 8 and 12) that when there is no asymptote involved, that the limit exists and hence the improper integral converges. This is NOT always the case, and it is always necessary to evaluate the limit explicitly than to make an assumption.

Example 14. *Calculate* $\int_0^3 g(x)\,dx$*, if possible, for the piecewise-defined function*

$$g(x) = \begin{cases} 1 & \text{if } x \le 2 \\ \frac{1}{(x-2)^2} & \text{if } x > 2 \end{cases}.$$

Note that here, as in the last example, the domain of $g(x)$ is all real numbers. However, there is a vertical asymptote for $g(x)$ at $x = 2$.

We start be separating out the discontinuity:

$$\begin{aligned}\int_0^3 g(x)\,dx &= \int_0^2 dx + \int_2^3 \frac{1}{(x-2)^2}\,dx \\ &= \int_0^2 dx + \lim_{c\to 2^+} \int_c^3 \frac{1}{(x-2)^2}\,dx\end{aligned}$$

since $g(x)$ is continuous from the right at $x = 2$. Continuing, we get

$$\begin{aligned}
\int_0^3 g(x)\,dx &= \int_0^2 dx + \lim_{c\to 2^+} \int_c^3 \frac{1}{(x-2)^2}\,dx \\
&= \left(x\Big|_0^2 \right) + \lim_{c\to 2^+} \left(-\frac{1}{(x-2)}\Big|_c^3 \right) \\
&= 2 + \lim_{c\to 2^+} \left(-1 + \frac{1}{c-2} \right) = 2 + \lim_{c\to 2^+} \left(\frac{3-c}{c-2} \right) = \infty.
\end{aligned}$$

This integral diverges even though part of it is fine.

Example 15. *For $a > 0$, calculate $\int_{-a}^{a} \frac{1}{\sqrt[5]{x}}\,dx$, if possible.* This integrand is an odd function. Hence it is symmetric with respect to the origin. Recall the the integral of an odd continuous function on the interval $[-a, a]$ should be 0. However, since the integral is improper (asymptote at $x = 0$), we do not know if the integral even converges. To check, we need to actually do the calculation.

Here we get

$$\begin{aligned}
\int_{-a}^{a} \frac{1}{\sqrt[5]{x}}\,dx &= \int_{-a}^{0} \frac{1}{\sqrt[5]{x}}\,dx + \int_{0}^{a} \frac{1}{\sqrt[5]{x}}\,dx \\
&= \lim_{c\to 0^-} \int_{-a}^{c} \frac{1}{\sqrt[5]{x}}\,dx + \lim_{c\to 0^+} \int_{c}^{a} \frac{1}{\sqrt[5]{x}}\,dx \\
&= \lim_{c\to 0^-} \left(\frac{5}{4}\sqrt[5]{x^4}\Big|_{-a}^{c} \right) + \lim_{c\to 0^+} \left(\frac{5}{4}\sqrt[5]{x^4}\Big|_{c}^{a} \right) \\
&= \lim_{c\to 0^-} \left(\frac{5}{4}\sqrt[5]{c^4} - \frac{5}{4}\sqrt[5]{(-a)^4} \right) + \lim_{c\to 0^+} \left(\frac{5}{4}\sqrt[5]{a^4} - \frac{5}{4}\sqrt[5]{c^4} \right) \\
&= -\frac{5}{4}\sqrt[5]{a^4} + \frac{5}{4}\sqrt[5]{a^4} = 0.
\end{aligned}$$

The thing here is that both of the limits actually do exist, and are the same magnitude yet opposite signs. Hence the origin integral converges for any value of $a > 0$, and the integral is 0.

Finding the value of a definite integral which is improper means evaluating a limit as well as using the Fundamental Theorem of calculus (using the antiderivative evaluated at the limits). However, many time, the actual value of the integral is not nearly as important as knowing whether the integral converges or not. This is particularly true when the integral is difficult or impossible to solve analytically. Fortunately, and like other areas of calculus, integrals behave well when compared to each other. We start with a major theorem:

Theorem 16. *Suppose $f(x)$, $g(x)$ are continuous with $f(x) \geq g(x) \geq 0$ for all $x \geq a$. Then if $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ converges. And if $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges.*

Notes:

- The improper integrals above represent unbounded areas between their respective curves and the x-axis. The integral of the larger function must be bigger than that of the smaller (there is more area). Thus if the integral of the larger is finite, the area of the smaller must also be finite. Also, if the area of the smaller is not finite, then neither can the area of the larger.
- This is also true is the two continuous functions cross each other a number of times, as long as at some point they stop crossing and one dominates the other in the "tail".
- really, this is ONLY an existence theorem, and says nothing about the actual value of either of the integrals, if they exist.

I then did example 9 from this section in detail. Also pay attention to Example 10.

1. Lecture 3: Sequences

On to sequences.

Definition 17. A *sequence* of numbers is simply a infinite list of numbers, denoted by a variable and a subscript that takes values in the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$,

$$a_1, a_2, \ldots, \qquad \text{or} \qquad \{a_n\} \qquad \text{or} \qquad \{a_n\}_{n \in \mathbb{N}} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}.$$

Notes:

- Sometimes, the subscript will take values other than the natural numbers. For example, it may start at 0, or may incorporate some of the negative integers. it will be obvious, however, from the context, what the range of values of the subscript will be in the application.
- Many times, the sequence follows a pattern which may be set out according to a rule (like a function) based on the subscript:
$$a_n = \frac{1}{n}, \qquad \text{or} \qquad b_n = \frac{(-1)^n}{2^n}, \qquad \text{or} \qquad c_n = \sin 2\pi n.$$
These sequences are actually functions of their subscript, and all can be written as $a_n = f(n)$. The first and last look like the integer points of a continuous function. For example, $a_n = f(n)$, where $f(x) = \frac{1}{x}$. This would be harder to do for the sequence $\{b_n\}$.

- Graphing a sequence is like graphing $y = f(x)$, except that the graph is a discrete set of points in the plane with coordinates (n, a_n).

Definition 18. A sequence $\{a_n\}$ has a limit L and we write

$$\lim_{n\to\infty} \{a_n\} = L, \quad \text{or} \quad \lim_{n\to\infty} a_n = L, \quad \text{or} \quad \{a_n\} \longrightarrow L$$

if, for every $\epsilon > 0$, there is a corresponding N where, if $n > N$, then $|a_n - L| < \epsilon$.

Here, the last inequality is not difficult to see. $|a_n - L| < \epsilon$ is the same as $-\epsilon < a_n - L < \epsilon$, which is the same as $L - \epsilon < a_n < L + \epsilon$. This is simply a small band around a number L of width ϵ is either direction (for a total width of 2ϵ). The definition says that given a sequence, the number L will be the limit of the sequence provide no matter how small we make the band, eventually the sequence will enter that band and never leave again. This is exactly the same as what you studied for function way back in Calculus I.

Really, having a limit or not is the ONLY interesting thing about sequences. And it is an important quality to have. So having the means to determine if a sequence has a limit and to be able to find it if it does have one are paramount. There are many ways to discover limits of sequences.

Use the function. Suppose $a_n = f(n)$ for some function $f(x)$ which is continuous on $[1, \infty)$. If $f(x)$ has a horizontal asymptote at ∞, then the sequence has no choice but to follow the function, and the sequence converges. In other words:

Proposition 19. *For $a_n = f(n)$, where $f(x)$ is continuous on $[1, \infty)$, if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.*

Example 20. Let $a_n = 2 - \frac{1}{n}$. Since $f(x) = 2 - \frac{1}{x} = \frac{2x-1}{x}$ is continuous on $[1, \infty)$, and $\lim_{x\to\infty} f(x) = 2$, it follows that $\lim_{n\to\infty} a_n = 2$ also.

Note: The converse is definitely NOT true. Just because a sequence defined by $a_n = f(n)$ converges, it does not follow that the original function has a horizontal asymptote at infinity!

Example 21. Let $f(x) = \cos 2\pi x$. $f(x)$ is continuous for all reals. And $f(x)$ does not have a horizontal asymptote at infinity. This means that $\lim_{x\to\infty} f(x)$ does not exist. It is the cosine function. But

$$a_n = f(n) = \cos 2\pi n = 1$$

for all $n \in \mathbb{N}$. This sequence does converge (to 1, that is). In fact, the sequence is just the peaks of all of the humps that is the graph of the cosine function.

Also, when using the continuous function $f(x)$ to evaluate the limit of a sequence $a_n = f(n)$, one can use all of the tools of calculus. But ONLY on the continuous function!

Example 22. *Does $a_n = n^2e^{-3n}$ have a limit?* Again, associate to this sequence the continuous function $f(x) = x^2e^{-3x} = \frac{n^2}{e^{3x}}$. Does $f(x)$ have a horizontal asymptote? Since $f(x)$ is not only continuous but differentiable, we can use all of our calculus tools to study this function. Evaluate $\lim_{x\to\infty} \frac{x^2}{e^{3x}}$. Here, if we push out x to infinity in the expression (in a sense, evaluating the limits of the top and bottom of the fraction), we get the indeterminate form $\frac{\infty}{\infty}$. Thus we can apply L'Hospital's Rule so that

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^{3x}} = \lim_{x\to\infty} \frac{2x}{3e^{3x}}.$$

Repeating the idea, we see again that we would get the indeterminate form $\frac{\infty}{\infty}$. Applying L'Hospital's Rule a second time gets us

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}} = \lim_{x\to\infty} \frac{\frac{d^2}{dx^2}x^2}{\frac{d^2}{dx^2}e^{3x}} = \lim_{x\to\infty} \frac{2}{9e^{3x}} = 0.$$

Hence a_n does indeed have a limit.

And now a practical word. One CANNOT use something like L'Hospital's Rule on the sequence a_n. The sequence is NOT differentiable.

More notes:

- if a limit of a sequence exists, we say that the sequence *converges.* if the limit does not exist, then the sequence *diverges.*
- Limits of sequences behave exactly like limits of functions and expression that you learned in an early chapter of the book. You should pay attention to all of the rules on page 678.

Example 23. Product Rule for Limits. Remember the Product rule for Limits: Suppose $\{a_n\}$, and $\{b_n\}$ are two convergent sequences (their limits exist!). Then

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$

But caution: This ONLY works when both $\{a_n\}$, and $\{b_n\}$ are convergent.

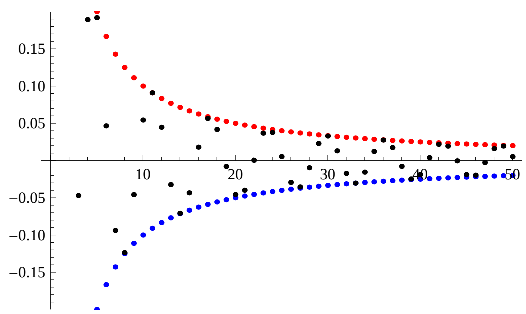
For example, let $\{a_n\} = \{n\}$, and $\{b_n\} = \left\{\frac{2}{n}\right\}$. Then

$$2 = \lim_{n\to\infty} n \cdot \frac{2}{n} \neq \left(\lim_{n\to\infty} n\right)\left(\lim_{n\to\infty} \frac{2}{n}\right)$$

since the second limit on the right is 0, while the first is ∞ (it does not exist). Hence the product rule FAILS when one of the individual sequences does not have a limit.

work with these limit rules and get very comfortable with them....

Squeezing sequences into limits. It should be quite clear that if one convergent sequence is term-by-term larger than another convergent sequence, then the limit of the first will be bigger then the limit of the second. We can take this one step further by saying that if one sequence lives term-by-term within two other convergent sequences, than the limit of the "sandwiched" sequence, if it exists, will live between the other two. The following theorem follows this theme to a very nice conclusion:



Theorem 24. *Suppose* $a_n \leq c_n \leq b_n$ *for* $n > n_0$, *and* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$. *Then* $\lim_{n\to\infty} c_n = L$.

Example 25. *Show the sequence* $\{c_n\} = \left\{\frac{\sin n}{n}\right\}$ *converges to* 0. This should not be hard to see as the numerator never ventures far from 0, while the denominator gets large as the sequence progresses. However, to actually show the result takes a bit of work. One can use the corresponding function $f(x) = \frac{\sin x}{x}$ and try to find an asymptote at infinity, as above. But L'Hospital's Rule doesn't apply here (why not?).

Instead, we will employ the Squeezing Theorem. To start, notice that the numerator satisfies $-1 \leq \sin n \leq 1$ (one of the nice things about this trig function). Divide the inequality by n to get

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

Let $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$, and we have

$$a_n \leq c_n \leq b_n$$

just as in the theorem (choose $n_0 = 1$ but any choice of n_0 will do). And since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$, the theorem concludes that also $\lim_{n\to\infty} c_n = 0$. Hence $\left\{\frac{\sin n}{n}\right\} \longrightarrow 0$. See the picture, where a_n is in blue on the bottom, and b_n is in red on top. Both are squeezing c_n in black (the more random-looking points in the middle).

We will continue next time....