110.109 CALCULUS II

Week 6 Lecture Notes: March 5 - March 9

Lecture 1

To study some of the properties of polar curves, we should understand how they work in relation to the rectilinear coordinates. Then we can use the concepts and formulas we learned there to re-engineer similar concepts that are uniquely polar. the best way to do that is to "view" a polar curve as simply an interesting parameterized curve. To this end, $r = f(\theta)$ be a polar curve and notice that the equations converting polar coordinates to rectilinear coordinates $x = r\cos\theta$ and $y = r\sin\theta$, are actually simply functions of θ when restricted to the polar curve:

$$x(\theta) = f(\theta)\cos\theta \text{ and } y(\theta) = f(\theta)\sin\theta.$$

Thus the polar curve is just a parameterized curve where θ is the parameter. So the calculation of the slope of the curve is basically the same as that of Section 10.2:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

While this looks much harder to calculate in general due to the trigonometric functions, the theory involves nothing more than this expression.

Example 1. Since it is always best to start with the obvious example, let $r = 1$ be our polar curve. The graph is the unit circle in the plane, and with our parameter equations, we have $x(\theta) = \cos\theta$ and $y(\theta) = \sin\theta$. Thus the slope of the tangent line to the polar curve is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta = \frac{\frac{-x}{r}}{\frac{y}{r}} = \frac{-x}{y}.$$

This is our original conclusion when we viewed the unit circle as $x^2 + y^2 = 1$.

Example 2. *Find the values of θ where the slope of the line tangent to $r = 4\cos\theta$ is horizontal and vertical.* Here $\frac{dr}{d\theta} = r\sin\theta$, so the slope of the tangent line at a value of θ is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-4\sin\theta\sin\theta + 4\cos\theta\cos\theta}{-4\sin\theta\cos\theta - 4\cos\theta\sin\theta}.$$

Finding the horizontal lines means setting this last expression to 0. Hence

$$-4\sin\theta\sin\theta + 4\cos\theta\cos\theta = 0$$
$$\sin^2\theta = \cos^2\theta,$$

which is solved precisely when either $\sin\theta = \cos\theta$, or $\sin\theta = -\cos\theta$. It turns out this happens rather often, when $\theta = (2n+1)\frac{\pi}{4}$, $n \in \mathbb{Z}$. This is any ODD integer multiple of $\frac{\pi}{4}$. But given the graph above, ANY odd multiple of $\frac{\pi}{4}$ lands on one of the two points at the top

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and bottom of the circle given by θ_1 and θ_2. See the graph. The two red lines correspond to $\theta = \pm\frac{\pi}{4}$. How would one go about finding the vertical tangents? Really, just set the denominator to 0 and solve. You should get $\theta = n\frac{\pi}{2}$, $n \in \mathbb{Z}$: The boiled-down equation is $\sin\theta\cos\theta = 0$.

Example 3. *Do the same for the Cartioid* $r = 1 + \sin\theta$*.* Here, the equations become more involved.

Example 4. *What is the equation of the line tangent to the polar curve given by* $\theta = c$*, a constant?*

We also spent some time talking about the exam on Wednesday.

Lecture 2: Midterm 1 day, no lecture

Lecture 3

Today, we discussed the last topic concerning curves in the plane using the system of polar coordinates; that of calculating the length of a curve directly within this coordinate system. Often, we can first switch back to rectilinear coordinates, and use a formula we already developed. But this may be difficult, or even impossible if the curve is not the graph of a function in either or both sets of coordinates. It turns out the best place to start is to think of a parameterized curve. Here is where we begin:

Let $r = f(\theta)$ be a polar curve. We know, given a function $y = F(x)$, that the length of a curve, from $x = a$ to $x = b$, is given by the formula

$$\text{Length } = \int_a^b \sqrt{1 + \left(F'(x)\right)^2}\, dx.$$

To review this, go to Section 8.1. This is part of the syllabus for the course 110.108 Calculus I. we also know that, given a parameterization of the same curve $x(t)$ and $y(t)$, that the curve is $y(t) = F\left(x(t)\right)$, that

$$\text{Length } = \int_a^b \sqrt{1 + \left(F'(x)\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\, \frac{dx}{dt}\, dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

where $x(\alpha) = a$ and $x(\beta) = b$. Note that this is really just a substitution type argument, and is what I call the Anti-Chain Rule. See last week's lectures for details, or Section 10.2.

We seek to play the same game for a polar curve. Indeed, for any polar curve $r = f(\theta)$, we can re-write the equations that relate polar coordinates back to rectilinear coordinates

$$x = r\cos\theta = f(\theta)\cos\theta) \text{ and } y = r\sin\theta = f(\theta)\sin\theta).$$

In this way, curve is written in the rectilinear coordinates and parameterized by θ. Then the formula for arc length comes directly from Section 10.2:

$$\text{Arc Length } = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

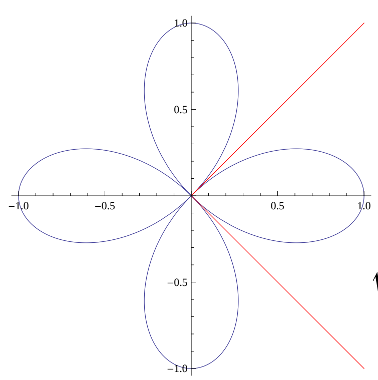
This is workable, but having to go back to rectilinear coordinates to calculate the length of a polar curve is not optimal. Better to be able to calculate directly, no? It turns out that we can. First, write out the derivatives of the parameterization:

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{d}{d\theta}(r\cos\theta) = \frac{dr}{d\theta}\cos\theta - r\sin\theta\\ \frac{dy}{d\theta} &= \frac{d}{d\theta}(r\sin\theta) = \frac{dr}{d\theta}\sin\theta + r\cos\theta.\end{aligned}$$

Note that we need the product rule here because r is a function of θ. Back to the formula for arc length, we place this in and look for simplifications. We get

$$\begin{aligned}\text{Arc Length } &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta\\ &= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}\, d\theta\\ &= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\cos^2\theta}\, d\theta\\ &= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2\left(\cos^2\theta + \sin^2\theta\right) + r^2\left(\sin^2\theta + \cos^2\theta\right)}\, d\theta\\ &= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta\end{aligned}$$

We now have a formula to calculate the length of a polar curve directly.



Example 5. *Set up the calculation to find the length of the perimeter of of one leaf of a 4-leaf rose* $r = \cos 2\theta$. Here, the 4-leaf rose has one of its petals lying symmetrically on the polar axis (the positive x axis in rectilinear coordinates). The integrand of the arc-length integral is

$$\begin{aligned}\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} &= \sqrt{(-2\sin 2\theta)^2 + (\cos 2\theta)^2}\\ &= \sqrt{4\sin^2 2\theta + \cos^2 2\theta} = \sqrt{3\sin^2 2\theta + 1},\end{aligned}$$

since $\frac{dr}{d\theta} = -2\sin 2\theta$, and we use the standard identity $\sin^2 x + \cos^2 x = 1$ to help simplify what is under the radical.

We still need to know where to integrate, though (where the limits are). Essentially, we need to find the interval of θ where the leaf is traced exactly once. In this case, it is fairly easy: Look for two consecutive places where $r = 0$. We can solve:

$$r = 0 = \cos 2\theta \iff 2\theta = \frac{\pi}{2} \iff \theta = \frac{\pi}{4}, \text{ and}$$
$$\iff 2\theta = -\frac{\pi}{2} \iff \theta = -\frac{\pi}{4}.$$

These are the two red lines in the figure. Our calculation is then

$$\text{Arc Length } = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{3\sin^2 2\theta + 1}\, d\theta$$

and we are done.

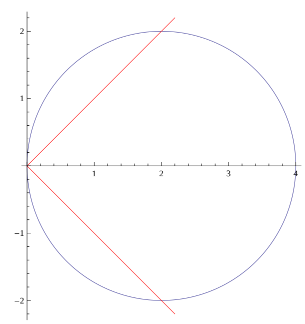
Example 6. *Find the length of the graph of the one-leaf rose $r = 4\cos\theta$.* Here, $f(\theta) = 4\cos\theta$, and $f'(\theta) = -4\sin\theta$. What are the limits of the arc-length integral? The interval of θ where the leaf is traced out once. Recall that the off-leaf roses are traced twice over the full range of θ. Here, any interval of length π will do. We choose $\theta = 0$ and $\theta = \pi$ for now. Our length then is

$$\text{Arc Length } = \int_0^{\pi} \sqrt{(4\cos\theta)^2 + (-4\sin\theta)^2}\, d\theta = \int_0^{\pi} \sqrt{16\cos^2\theta + 16\sin^2\theta}\, d\theta = \int_0^{\pi} \sqrt{16}\, d\theta = 4\theta\Big|_0^{\pi} = 4\pi.$$

So what is the perimeter of a circle of radius 2 (see figure below)?

Today we started Section 7.8, on Improper Integrals. To start the discussion, consider the integral

$$\int_1^2 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}.$$



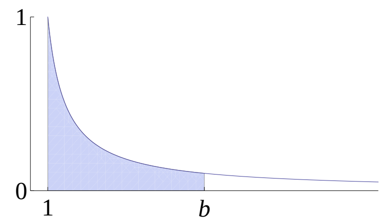
This is simply a straightforward application of the Anti-Power Rule, and since the function is positive for all $x > 0$, the definite integral represents the area between the curve $f(x) = \frac{1}{x^2}$ and the x-axis. if we changed the upper limit from 2 to a 5, or to a 10, or to a 1000, how would the calculation change? Indeed, for ANY $b > 0$ (including $0 < b < 1$, we get

$$\int_1^b \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^b = -\frac{1}{b} + 1 = \frac{b-1}{b}.$$

Looking at this form, answer the following question:

Question. *Is there a $b > 1$ where $\int_1^b \frac{1}{x^2}\, dx > 1$?*

The answer, by looking at the form $\frac{b-1}{b}$, is most certainly not. Then can we say something about whether the following limit exists and what it is (if it exists):



$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx.$$

Indeed, we can. Since for any $b > 1$, the integral is a definite integral of a positive function, the value of the definite integral, as the area of the region bounded by the function and the x-axis on the interval $[1, b]$ is positive. And since the integral can be expressed simply as a expression involving only b, we can then evaluate the limit using techniques from back in Calculus I:

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{x}\bigg|_1^b\right) = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = \lim_{b\to\infty}\frac{b-1}{b} = 1.$$

Note. *This means the the area of the* unbounded *region represented between the function and the x-axis on the infinite interval* $[1,\infty)$*, is bounded and equal to 1.*

Really what we are asking for is the quantity

$$\int_1^\infty \frac{1}{x^2}\,dx,$$

but the limit does not make sense given our rules for integration. It is for this reason that we call such an integral an *improper integral*, and define it as such:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx.$$

Definition 7. For $f(x)$ continuous for all $x \geq a$, the improper integral

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx,$$

provided the limit exists. If the limit exists, we say the improper integral *converges*. Else, we say it *diverges*.

The effect of this calculation is that one calculates the definite integral for some b as the upper limit, giving an area (if the function is positive. Else the value of the integral loses it's interpretation as an area.) in terms of b. Then we push b to infinity and watch how the value changes. If the value-changes settle down to a number (in the limit), then that number IS the value of the improper integral. There are many examples, some of which I gave in class, and many more in the book. Here is one.

Example 8. Calculate $\int_1^\infty \frac{1}{x}\,dx$, if it exists. Here

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\ln|x|\right)\Big|_1^b = \lim_{b\to\infty}\left(\ln|b| - \ln|1|\right) = \lim_{b\to\infty}\ln b = \infty,$$

hence does not exist (the improper integral diverges).

We will continue with more examples next time.