110.109 CALCULUS II

Week 5 Lecture Notes: February 27 - March 2

LECTURE 1

Today, we start with an example of how to use a parameterization to calculate the area between a parameterized curve and the x-axis.

Example 1. Calculate the area between the upper half circle and the x -axis where the circle is the standard unit circle centered at the origin in the plane. This is the circle defined by the equation $x^2 + y^2 = 1$. Note that we now have three ways to calculate this area:

(1) **Geometry.** The area inside a circle of radius r is $A = \pi r^2$. Here $r = 1$, and we are looking for the area of precisely half the circle, so
$$A = \frac{1}{2}\pi r^2\Big|_{r=1} = \frac{\pi}{2}.$$

(2) **Standard integration.** The upper half-circle, to the x-axis, precisely bounds the region we seek to calculate the area of. And on this region, we can solve the original equation defining the circle for y: $y = \sqrt{1-x^2}$. Thus we have
$$A = \int_{-1}^{1} \sqrt{1-x^2}\, dx.$$
We can go back to Chapter 7 for a nice technique to solve this integral. Let $x = \sin\theta$, then $dx = \cos\theta\, d\theta$. This trigonometric substitution will help us to "process" the radical in a way that allows for an easier calculation. Changing the limits also, we get $1 = x = \sin\theta$ is solved for $\theta = \frac{\pi}{2}$, and for $x = -1$, the obvious choice would be $\theta = \frac{3\pi}{2}$. This choice would be fine, although you would be creating a lower limit that is larger than your upper limit. This is not a problem, really, as you can simply switch the limits and add a negative sign to the integrand (having a lower limit greater than the upper limit effectively means integrating backwards. That is where the minus sign comes from). A more natural choice, though is to realize that $x = -1$ also corresponds to $\theta = -\frac{\pi}{2}$. With this choice, we would be integrating in from a lower *theta* to a high one. No additional processing needed. Then
$$\begin{aligned} A = \int_{-1}^{1} \sqrt{1-x^2}\, dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\cos\theta\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos\theta|\cos\theta\, d\theta. \end{aligned}$$
On the interval $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta \geq 0$. Hence the absolute value signs are not needed, and
$$\begin{aligned} A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos\theta|\cos\theta\, d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}\left(-\frac{\pi}{2}\right) + \frac{1}{4}\sin 2\left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2}. \end{aligned}$$

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(3) **Using the parameterization.** Here, we will use another parameterization of the circle that we saw in a previous lecture. Let $x(t) = \sin 4\pi t$, and $y(t) = -\cos 4\pi t$. Then the circle is fully parameterized for t from $t = 0$ to $t = \frac{1}{2}$. (Why?) Here again, we will need to understand the limits of the integral in terms of t and not x. For $x(t) = \sin 4\pi t = 1$, we will need $4\pi t = \frac{pi}{2}$, so that $t = \frac{1}{8}$. For $x = -1$, we get $t = \frac{3}{8}$. Not worrying about the size of these limits, we now calculate:

$$A = \int_{-1}^{1} \sqrt{1-x^2}\, dx = \int_{\frac{3}{8}}^{\frac{1}{8}} y(t)x'(t)\, dt = \int_{\frac{3}{8}}^{\frac{1}{8}} (-\cos 4\pi t)\,(4\pi \cos 4\pi t)\ dt.$$

Realizing that the limits should be reversed, adding a negative sign to the integrand, we finish the calculation:

$$\begin{aligned} A = \int_{\frac{3}{8}}^{\frac{1}{8}} (-\cos 4\pi t)\,(4\pi \cos 4\pi t)\ dt &= \int_{\frac{1}{8}}^{\frac{3}{8}} 4\pi \cos^2 4\pi t\, dt \\ &= \int_{\frac{1}{8}}^{\frac{3}{8}} 4\pi \left(\frac{1}{2} + \frac{1}{2}\cos 8\pi t\right)\ dt = \left[\frac{4\pi}{2}t + \frac{4\pi}{16\pi}\sin 8\pi t\right]\Big|_{\frac{1}{8}}^{\frac{3}{8}} \\ &= \left(\frac{4\pi}{2}\frac{3}{8} + \frac{4\pi}{16\pi}\sin 8\pi\left(\frac{3}{8}\right)\right) - \left(\frac{4\pi}{2}\frac{1}{8} + \frac{4\pi}{16\pi}\sin 8\pi\left(\frac{1}{8}\right)\right) \\ &= \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}. \end{aligned}$$

Note. *It does not matter which parameterization you choose for this calculation. As long as you transform your limits correctly, and interpret the direction of integration correctly, the answer will be the same.*

Two more points of interest in this section. The first is that one can calculate the surface area of certain objects formed as "surfaces of revolution" using the calculus you learned in calculus I: Given a function $y = F(x)$, where on an interval $x \in [a, b]$, $F(x) \geq 0$, the surface area of the object formed by "rotating" the graph of $F(x)$ around the x-axis (out of the page, around and back through the page), is

$$SA = \int_a^b 2\pi F(x)\sqrt{1 + [F'(x)]^2}\, dx.$$

This should make sense, no? The area of any surface really is the product of a measurement of its dimensions. In this case, one can characterize these dimensions at any point as the circumference of the circle passing through the point (the $2\pi r$ part, where the radius if $F(x)$) times the length of the object (given by the length of the graph of $F(X)$. Integrating this over the interval $[a, b]$ gives the surface area. Translating directly into a parameterization of the curve given by $x(t)$ and $y(t)$ where $x(\alpha) = a$ and $x(\beta) = b$, we get

$$SA = \int_\alpha^\beta 2\pi F(x(t))\sqrt{1 + [F'(x(t))]^2}\frac{dx}{dt}\, dt = \int_\alpha^\beta 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

where $y(t) = F(x(t))$, and under the radical we employed the same sort of simplification we use before when calculating the length of a curve.

And secondly, recall that for an equation defining y implicitly as a function of x, we can both implicitly differentiate to find $\frac{dy}{dx}$, as well as use the parameterization. We got

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

But what if we wanted to also know how the derivative is changing with respect to x or t. What if we wanted to know $\frac{d^2y}{dx^2}$? The knee-jerk reaction would be to directly relate it to the second derivatives of the parameterization. But try it for any example. You will find that

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}. \tag{1}$$

Example 2. For the curve $x^2 + y^2 = 1$, calculate $\frac{d^2y}{dx^2}$, using the parameterization $x(t) = \cos t$, $y(t) = \sin t$.

By the "old fashioned way", we already know that $\frac{dy}{dx} = -\frac{x}{y}$ from a previous calculation (see how the derivative is also a function of both x and y, where y is an implicit function of x?). Thus, differentiating again, we get

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{-x}{y}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{-\cos t}{\sin t}\right)}{\frac{dx}{dt}} = \frac{\csc^2 t}{-\sin t} = \frac{-1}{\sin^3 t} = -\frac{1}{y^3}.$$

Now, assuming the *mistaken* idea in Equation 1 is TRUE, we could try to use the parameterization and get

$$-\frac{1}{y^3} = \frac{d^2y}{dx^2} \stackrel{?}{=} \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} = \frac{-\sin t}{-\cos t} = \tan t.$$

Here, $-\frac{1}{y^3} = -\csc^3 t$ is definitely NOT equal to $\tan t$! What is wrong is our point of view. Next time, we will correct this.

LECTURE 2

To continue, we need to understand better what we are doing. If y is an implicit function of x, then

$$\frac{d}{dx}(y) = \frac{\frac{d}{dt}(y)}{\frac{dx}{dt}}.$$

Once we calculate $\frac{dy}{dx}$ using implicit differentiation, realize that the answer $\frac{dy}{dx}$ will be equal to an expression also involving both x and y, where y is an implicit function of x. Hence also $\frac{dy}{dx}$ IS an implicit function of x!

Hence the correct calculation is to treat it just like y, and write

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

This makes sense, since the only strange part is the numerator in the fraction on the right. But since both x and y are function of t, $\frac{dy}{dx}$ is also a function of t (being a function of x and y). Hence it's all good.

Example 3. Again, let $x^2 + y^2 = 1$ be an equation of a curve in $\mathbb{R}^2$, parameterized by $x(t) = \cos t$ and $y(t) = \sin t$ on the $t \in [0, 2\pi]$. Calculate $\frac{d^2y}{dx^2}$.

Again, using $\frac{dy}{dx} = -\frac{x}{y}$ from the previous calculation, we get

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{-x}{y}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{-\cos t}{\sin t}\right)}{\frac{dx}{dt}} = \frac{\csc^2 t}{-\sin t} = \frac{-1}{\sin^3 t} = -\frac{1}{y^3}.$$

Now we can use the idea of a parameterization to introduction another coordinate system on the plane. There are many coordinate systems on the plane and really, all of them are identical in that they provide position for points in the plane, and allow us to use the coordinates to analyze the functional properties of relationship that we define on them. Today we will define a new coordinate system particularly good to locate position with regard to a single point, again the origin, called the pole. This system measures distance from the pole as one of its measurements and the angle from some reference ray emanating from the pole to locate direction. Think of how a radar or sonar system works, and you find the polar coordinate system: We will call these new coordinates

- r - range: the distance from the origin, measured in a straight line joining the point to the pole, and
- θ - angle: The angle, in radians, the ray joining the pole with the point makes with respect to the polar axis (for convention's sake, the polar axis is usually the positive x-axis.

This gives us a natural set of valid values for these two coordinates. Namely, $r \geq 0$, and $0 \leq \theta < 2\pi$. What happens when these coordinates take values outside of these values? This will take a bit of analysis, but really, it is necessary to know since we will be defining functions between these coordinates, and need to understand well what happens if a coordinate spills outside of these bounds.

There are a few ways that this new coordinate system needs to be studies to be well understood.

(1) The ONLY point which is zero distance from the origin (the pole) IS the pole. This point has coordinates $(r, \theta) = (0, 0)$, but also ANY angle will work here: Hence

$$\text{the origin or pole has coordinates } (r, \theta) = (0, \theta), \quad \text{for all } \theta.$$

If I am standing at the origin, it will not matter at what angle I am facing. I am still at the origin.

(2) The coordinate θ, being an angle, is 2π-periodic. This means that for any angle θ, we have $\theta + 2\pi = \theta$. Hence for points in the plane, we get

$$(r, \theta) = (r, \theta + 2n\pi), \text{ for any } n \in \mathbb{Z}.$$

(3) It will become clear that at times, a negative value of r will arise in a calculation. While *a priori* this does not make sense (a negative distance from the origin?!?), there is a valid way to define such a concept that maintains the integrity of the coordinate system: A negative value of r in a point's coordinates means that one measures out from the pole in the OPPOSITE direction than that specified by θ. Or

$$(-r, \theta) = (r, \theta + \pi).$$

Like in the rectangular coordinates system (x, y), we chose a coordinate to represent the independent variable (the one we have control over choosing values for). The other played the role of the dependent variable is defined functional relationships (this is the coordinate we want to study). We will do the same thing here, and the convention is to choose θ as the independent variable, and define r as dependent to θ. Thus our functions will look like $r = f(\theta)$. Note that this is arbitrary, and we could do things the other way, but as a convention, we will do it this way.

So let's look at some very basic functions. First, the constant functions and then something more intrest-ing:

Example 4. $r = f(\theta) = a$, **where** $a \in \mathbb{R}$. For each and every value of θ, the value of r does not change, and the graph of this function is a circle of points, all of which are the distance a from the origin. This is like the analog of the constant function $y = a$ in the rectilinear coordinate system, and is a kind of "horizontal line" of the polar coordinate system. However, the graph is not a straight line. Note that for negative values of a, the graph is the same as that for the positive value, and $r = -a$ has the same graph as $r = a$.

Example 5. $\theta = a$. Here the graph is the set of all points that satisfy (r, a) in the plane. It should be easy to see that this set is a straight line through the pole, at angle θ from the polar axis. The negative values for r account for the line passing through the origin and continuing on the "other side". This is NOT the graph of a function $r = f(\theta)$, as it would fail the "vertical line test" miserably. But it is the graph of a set of an equation involving r and θ. Note that this is much like the "vertical" line in the rectilinear coordinate system given by $x = a$, except that the vertical lines in the polar coordinate system all pass through the origin.

Example 6. $r = \sin\theta$**.** Tying r to θ via a trigonometric function is a sure way to create beautiful graphs. The graph winds us being a perfect circle, though not one centered at the origin. Following its values as θ goes from 0 to π, the entire graph is drawn. Then, as θ goes from π to 2π, the sine function is negative. Thus the graph includes points a positive distance from the origin and at angle $\theta + \pi$. But these angles have already been covered by the graph. It turns out that the graph retraces itself precisely twice on the full range of values from $\theta = 0$ to $\theta = 2\pi$. This certainly is NOT always the case, but it is here. This also will be vitally important to know later when we start doing calculus on these functions.

There are more examples, which I showed in class on the computer, and placed on the course website and pointed to on the Facebook page. Enjoy them, play with them.

Lecture 3

For today, let's try to understand better why the graph of the function $r = \sin\theta$, is a perfect circle, although NOT centered at the pole (the origin). Realize that you are quite accustomed to thinking in the standard x and y rectilinear coordinates, and the polar coordinates are a new way to parameterize the plane $\mathbb{R}^2$. As the r and θ coordinates are a way to re-parameterize the plane, there should be a way to go from one to the other.

To start, realize that for any right triangle, there are natural ways to relate the magnitude of one of the non-right angles to the lengths of the sides. This IS the trigonometry you studied pre-calculus. In any right triangle, choose a reference vertex (not the right-angle one). Then angle θ at that vertex has the hypotenuse as one of its edges. If the hypotenuse has length r, then the other two sides have length $x = r\cos\theta$ (the adjacent side to the vertex), and $y = r\sin\theta$ (the opposite side to the vertex). But these are precisely the rectilinear coordinates (x, y) of the point (r, θ) for any point in the first quadrant when the pole sits on the origin of the rectilinear coordinate system. Generalizing to point in all four quadrants is straightforward, and hence knowing (r and θ, we can recover x and y. How to go backward? First, realize that the Pythagorean Theorem also relates the three sides of a right triangle: $r^2 = x^2 + y^2$. Hence, we can solve for the positive value of r as $r = \sqrt{x^2 + y^2}$. And for the angle, we also know that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}.$$

Hence we get $\theta = \arctan\left(\frac{y}{x}\right)$. Although this may be hard to calculate, we can now also go from rectilinear points to polar points.

Example 7. Let's go back to $r = \sin\theta$. Changing coordinates back to rectilinear coordinates, we get

$$\begin{aligned} r &= \sin\theta \\ r^2 &= r\sin\theta = y \\ x^2 + y^2 &= y \\ x^2 + y^2 - y &= 0 \\ x^2 + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4} &= 0 \qquad \text{(We are completing the square here.)} \\ x^2 + \left(y - \frac{1}{2}\right)^2 &= \frac{1}{4} = \left(\frac{1}{2}\right)^2. \end{aligned}$$

Thus, the equation $r = \sin\theta$ has the same graph as that of the last equation above. We know the one above as the circle of radius $\frac{1}{2}$, centered at the point $\left(0, \frac{1}{2}\right)$. Check the graph now.

Next time, we will begin the process of analyzing functions using some of the same tools we used in Calculus I for the rectilinear coordinate system. We will find that our knowledge of parameterizations will be very useful here.