110.109 CALCULUS II

Week 4 Lecture Notes: February 20 - February 24

LECTURE 1

Today, I decided to go over some of the content of last Friday's lecture, given by our own Recitation Instructor Kalina Mincheva. The topic was Section 9.5 Linear Differential Equations. The most general form for a linear first-order differential equation is

$$R(x)y' + S(x)y + T(x) = 0.$$

We call such a differential equation linear since it is linear in the dependent variable (the unknown function $y = y(x)$), and its derivative y'. Recall that "linear" basically means that the variable appears with exponent 1 in each monomial, is not the inside function of any composition of functions, and not multiplied in any of the other variables considered part of the "linear" set of variables. Thus, we couldn't have expressions like yy', or $\sin(y)$ in a linear ODE.

Note. *The three functions of x above, $R(x)$, $S(x)$, and $T(x)$, can be ANY function of x, linear or not. The ODE is called linear if it is linear in the* dependent *variable and its derivatives.*

In this case, the "Standard Form" for a first order linear ODE is

$$y' + \frac{S(x)}{R(x)}y + \frac{T(x)}{R(x)},$$

or

$$y' + P(x)y = Q(x), \tag{1}$$

where $P(x) = \frac{S(x)}{R(x)}$, and $Q(x) = \frac{-T(x)}{R(x)}$. None of the particulars of this is really important, except that this is the form the section starts with as the form for a first-order linear differential equation.

The integrating factor is given by $e^{\int P(x)\,dx}$, and multiplying the ODE (Equation 1) through by this factor yields

$$e^{\int P(x)\,dx}y' + e^{\int P(x)\,dx}P(x)y = e^{\int P(x)\,dx}Q(x).$$

The left-hand side is the one to watch here, as the sum of the two terms is actually just the total derivative of the product of the unknown function y and the integrating factor. This works this way *precisely* due to the constructi0on of he integrating factor. Note explicitly that

$$\begin{aligned}\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] &= \frac{d}{dx}\left[e^{\int P(x)\,dx}\right]y + e^{\int P(x)\,dx}y' \\ &= e^{\int P(x)\,dx}\frac{d}{dx}\left[\int P(x)\,dx\right]y + e^{\int P(x)\,dx}y' = e^{\int P(x)\,dx}P(x)y + e^{\int P(x)\,dx}y'.\end{aligned}$$

Hence

$$\begin{aligned}e^{\int P(x)\,dx}y' + e^{\int P(x)\,dx}P(x)y &= e^{\int P(x)\,dx}Q(x) \\ \frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] &= e^{\int P(x)\,dx}Q(x).\end{aligned}$$

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Integrating both sides of this equation with respect to x (recall that when two functions of x are declared equal, their antiderivatives are also equal up to a constant), we get

$$\begin{aligned}\int \frac{d}{dx}\left[e^{\int P(x)\,dx}y\right]\,dx &= \int e^{\int P(x)\,dx}Q(x)\,dx \\ e^{\int P(x)\,dx}y &= \int e^{\int P(x)\,dx}Q(x)\,dx + C \\ y &= e^{-\int P(x)\,dx}\left[\int e^{\int P(x)\,dx}Q(x)\,dx + C\right] = e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx}Q(x)\,dx + Ce^{-\int P(x)\,dx}.\end{aligned}$$

Now, all of this really only means the following: Theoretically, it is always possible to SOLVE a first-order linear differential equation (Equation 1) by simply integrating. The fact that the integration may be hard or impossible notwithstanding. In practice, however, things get way more straightforward.

Example 1. Solve the differential equation $y' = x + y$.

Proof. This ODE is indeed linear, and can be placed in its standard form $y' - y = x$, where $P(x) = -1$ and $Q(x) = x$. The integrating factor is then

$$e^{\int P(x)\,dx} = e^{-\int\,dx} = e^{-x}.$$

Multiplying through the ODE in standard form, we get

$$\begin{aligned}e^{-x}\left[y' - y \,=\, x\right] \\ e^{-x}y' - e^{-x}y = e^{-x}x \\ \frac{d}{dx}\left[e^{-x}y\right] = xe^{-x}.\end{aligned}$$

This last equation looks fairly tame and easy to integrate. Now we integrate both sides with respect to x to get

$$\begin{aligned}\int \frac{d}{dx}\left[e^{-x}y\right]\,dx &= \int xe^{-x}\,dx \\ e^{-x}y &= -(x+1)e^{-x} + C.\end{aligned}$$

Note that you will need the method Integration by Parts to do the right-hand-side integral. Do this on your own! Finally, solve for the unknown function $y(x)$ by dividing by the always non-zero term e^{-x} to isolate y, we get

$$y(x) = -(x+1) + Ce^{x}.$$

To finish this problem, note the structure of the solutions. When $c = 0$, the solution is simply the line $y = -x - 1$. When $c \neq 0$, the solution is a sum of a line with an exponential. When the exponential is small (for negative values of x), the solution will look pretty much like a line. When the x values are large, the solution will look pretty much like an exponential. Now go into JODE and plot out the slope field. Does the slope field make sense vis a vis this solution? □

Example 2 (Example not from class)**.** Solve the differential equation $x^2y' - y = 2x^3e^{-\frac{1}{x}}$.

Proof. Note again that this ODE is definitely linear. To see this, place it in its standard form by dividing out by the term x^2 attached to y' (really, this means that at the top of this lecture, the function $R(x) = x^2$. Thus we get

$$y' - \frac{1}{x^2}y = 2xe^{-\frac{1}{x}},$$

so that the function $P(x) = -\frac{1}{x^2}$ in the theory above. This gives us an integrating factor of

$$e^{\int P(x)\,dx} = e^{-\int \frac{1}{x^2}\,dx} = e^{\frac{1}{x}}.$$

Multiplying through the ODE in standard form, we get

$$\begin{aligned} e^{\frac{1}{x}}\left[y' - \frac{1}{x^2}y \,=\, 2xe^{-\frac{1}{x}}\right] \\ e^{\frac{1}{x}}y' - e^{\frac{1}{x}}\frac{1}{x^2}y = e^{\frac{1}{x}}2xe^{-\frac{1}{x}} \\ \frac{d}{dx}\left[e^{\frac{1}{x}}y\right] = 2x. \end{aligned}$$

Note how much the last differential equation has been simplified. With problems contrived to explain a concept (like in the classroom), this happens often, no? The rest is simply calculus: Integrate both sides with respect to x to get

$$\begin{aligned} \int \frac{d}{dx}\left[e^{\frac{1}{x}}y\right]\,dx &= \int 2x\,dx \\ e^{\frac{1}{x}}y &= x^2 + C. \end{aligned}$$

Dividing by the always non-zero term $e^{\frac{1}{x}}$ to isolate y, we get

$$y(x) = \left(x^2 + C\right)e^{-\frac{1}{x}}.$$

Now, does this work? To check, note that using the solution we just calculated,

$$y'(x) = 2xe^{-\frac{1}{x}} + \left(x^2 + C\right)e^{-\frac{1}{x}}\left(\frac{1}{x^2}\right),$$

so that

$$\begin{aligned} y' - \frac{1}{x^2}y &= 2xe^{-\frac{1}{x}} \\ \left(2xe^{-\frac{1}{x}} + \left(x^2 + C\right)e^{-\frac{1}{x}}\left(\frac{1}{x^2}\right)\right) - \frac{1}{x^2}\left(\left(x^2 + C\right)e^{-\frac{1}{x}}\right) &= 2xe^{-\frac{1}{x}} \\ 2xe^{-\frac{1}{x}} + e^{-\frac{1}{x}} + \frac{C}{x^2}e^{-\frac{1}{x}} - e^{-\frac{1}{x}} - \frac{C}{x^2}e^{-\frac{1}{x}} &= 2xe^{-\frac{1}{x}}, \end{aligned}$$

which boils down perfectly. Hence we are done with this example. $\square$

Really, besides following this recipe for linear first-order differential equations, there is little more to do here. I then move into Section 10.1 on Parametric Equations. Noting that graphs of functions $y = f(x)$ always satisfy the Vertical Line Test (remember this?), I noted that many curves in the xy-plane cannot be expressed as functions in this way. This is because the curve does not satisfy the test. For examples, I gave $y^2 = x$, as well as $y^3 - y = x$, two functions that can be expressed as $x(y)$. Another stereotypical example is $x^2 + y^2 = 1$, the equation whose graph is the unit circle in $\mathbb{R}^2$ (the circle of radius 1 centered at the origin of the plane). In this case, attempting to solve for y, we get $y = \pm\sqrt{1 - x^2}$, which is NOT a functional relationship. Note that the part of the graph which depicts the "upper" semicircle (the part lying over the x-axis in the plane) can be expressed a function where $y(x) = \sqrt{1 - x^2}$. The same is true for the "lower" semicircle $y(x) = -\sqrt{1 - x^2}$. This is always true is one were to grab a part of the graph of a complicated curve which does satisfy the vertical line test, even if one cannot actually write the expression explicitly. As an example, try solving for either variable as a function of the other (even for a piece of the graph in the equation $y + \sin y = x + e^x$. Best to leave this equation as a way to describe y as an *implicit* function of x.

A technique to explore more general curves in the plane is to *parameterize* them: give them a parameter t and express the planar coordinates x and y as functions of the new variable t.

Example 3. For ANY functional relationship $y = f(x)$, one can always simply let $x(t) = t$, and then $y(t) = f(t)$. This gives you no real insight except that the independent variable x actually acts as a parameter on the curve in a functional relationship. It's values uniquely determine every point on the curve.

Example 4. For functions of the form $x(y)$, where x is a function of y (as in the cases $y^2 = x$ and $y^3 - y = x$ above, it is y that can play the role of the parameter, and we can write $y = t$, and $x(t) = t^3 - t$ (or leave y as the parameter.

What you get in all of these cases is the ability to choose values of the parameter to denote individual points ON the curve. For a more general example:

Example 5. For $x^2 + y^2 = 1$, try $x(t) = \cos t$, and $y(t) = \sin t$. For any value for $t \in \mathbb{R}$, we get

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1,$$

and we stay on the curve. Thus each value of t specifies a point $(x(t), y(t))$ in the plane (lying ON the curve).

Notes:

- In this way, we are assigning the curve (the circle here) its OWN coordinate t, and expressing x and y as functions of t. This is like taking a piece of string and putting ruler marking on it to denotes the points of the string, then placing the string in ANY configuration we like in the plane. The string still has its markings (values for t) and we can match the xy-coordinates of the string in the plane to these t markings.
- The parameterization we gave the circle in the example is NOT unique. Try this: Let $x = \sin 4\pi t$, and $y = -\cos 4\pi t$. Now mark off values for t on two graphs of the circle according to these two parameterizations. You will find that in (1) $t = 0$ corresponds to the point farthest to the right on the circle, $t = \frac{\pi}{2}$ corresponds to the top, and so on, and for (2) $t = 0$ corresponds to the bottom of the circle, $t = \frac{1}{8}$ corresponds to the right-most point, and so on.
- Given first a parameterization (x and y as functions of t), one can sometimes re-find the implicit equation between y and x by attempting to solve one of the function for t and subbing back into the other. This requires a bit of cleverness at times, but can be useful. For example, let $x(t) = 1 + \sqrt{t}$, and $y(t) = t^2 - 4t$. Then in the first equation, we get $(x - 1)^2 = t$, so that we wind up with
$$y = \left((x-1)^2\right)^2 - 4(x-1)^2.$$
- Parameterizations sometimes retrace their steps. for example, in $x^2 + y^2 = 1$, under the parameterization $x(t) = \cos t$, $y(t) = \sin t$, as t goes from 0 to 2π, every point on the circle corresponds to only one value of t. But as t gets larger, the increasing values for t retrace the circle over and over, in a periodic way, of period 2π. In the other parameterization of the circle above, the period is $\frac{1}{2}$ (Can you see this?).
- Parameterized curves sometimes look like pieces of graphs, even though the parameterization works for all $t \in \mathbb{R}$. For example, let $x(t) = \sin t$, and $y(t) = (\sin t)^2$. Think about this (I showed this on the overhead in class), but here $x^2 = y$ if we recreate the functional relationship between y and x. So the parameterized curve will live on the parabola. But since $-1 \leq \sin t \leq 1$, we also know that $-1 \leq xt \leq 1$, and $0 \leq y \leq 1$. Thus the parameterized curve will live on the parabola $y = x^2$, but only on the part of it that live in the square $-1 \leq xt \leq 1$, and $0 \leq y \leq 1$. I showed this in class and showed how a bead would travel on the parameterized curve for slowing varying values of t. The

path of the bead retraces the curve over and over again and is 2π-periodic, moving across the curve like a pendulum.

Example 6. For another example, let $x(t) = t^2 - 2t$, $y(t) = t+1$. Can you describe the curve in the xy-plane? Can you "see" the equation in the variables x and y. Whenever you can take a parameterization like this one, and solve one of the functions for the parameter t in terms of the other, you can then simply substitute back into the other equation to create the equation involving x and y (thus removing the parameter). Like above, we can do the following: $y = t + 1$ can be rewritten $t = y - 1$, and then

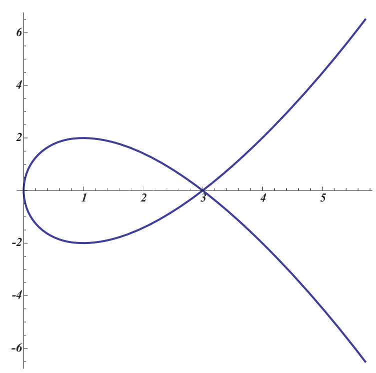
$$x = t^2 - 2t = (y-1)^2 - 2(y-1) = y^2 - 4y + 3.$$

Can you now see what the original curve will look like? What are its x and y intercepts? The shape is a parabola (why?). Where is its vertex? What is its axis of symmetry. Now draw the curve and label some of the t-parameter values ON the curve.

I then showed some of the more beautiful parameterizations like the Lissajou figures that are in the tan box in Section 10.1. We will continue next time.

LECTURE 2

Continuing the previous points,



- Curves in the plane may self-cross in way in which the parameterization may include points that correspond to two or more values of t. This may be a bit different that retracing a curve. Consider the example of $x(t) = t^2$, and $y(t) = t^3 - 3t$. The graph, at right, is also in the book. Notice that for $t = 0$, we are at the origin in the xy-plane. And for $t = \pm\sqrt{3}$, we are at the point $(3, 0)$. These are two distinct values for t, and we took two distinct paths along the curve from the origin to this point to get there. Say, we wanted to go along the curve from the origin at $t = 0$ to $t = \sqrt{3}$. In terms of increasing t along the path, did we travel above the x-axis along the curve or below it? Figure this out. The trick is to look at the y coordinate of the parameterization. Note that this gives the curve parameterized by t a direction of increasing t. The path has an "orientation" on it given by the parameterization.
- Parameterizations can certainly have corners and/or places where the curve stops. An interesting example of this (besides the one I gave $x(t) = \sin t$, and $y(t) = \sin^2 t$ at $t = \frac{\pi}{2}$) is the *cycloid*, also in the book, where $x(t) = t - \sin t$, and $y(t) = 1 - \cos t$. As we will see, the curve will not be differentiable here, although the parameterization actually IS....
- Sometimes even a parameterization has a parameter. Let $x(t) = t - a$, and $y(t) = t$, where $a \in \mathbb{R}$. For a choice of a, this is a curve (what does this curve look like?). As we vary a, how does the curve change? Now try parameterizing $x^2 + y^2 = a^2$ for different values of a. This amounts to a parameterized "family" of curves, each curve parameterized by t and the family parameterized by a.

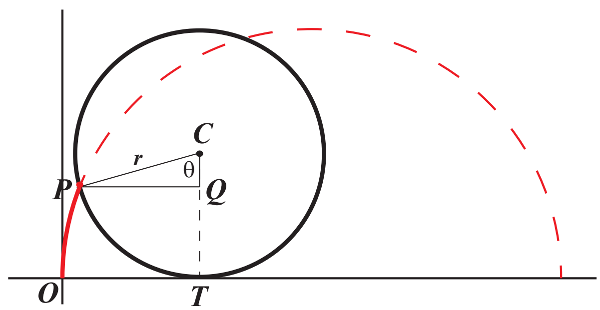
Example 7 (The Cycloid)**.** Fix a circle of radius r centered at the point $(0, r)$ in the xy-plane, and call P the point on the circle at the origin. Allow the circle to roll along the positive x-axis and follow the path that P traces in the plane. Describe this path in parametric equations.

Every point on a circle of radius r can be described by the trig functions $r\sin\theta$, and $r\cos\theta$, where θ is the angle the point makes with respect to some choice of reference radial line. This encourages us to use the variable θ as our parameter instead of some as yet undefined t.

First, note that if the circle rolls along the x-axis, the point where the circle touches the x-axis along its path is exactly equal to the arc-length of the circle as the circle rolls through the angle θ. This arc-length is proportional to the angle, and the proportional constant is the radius. Thus, in the drawing, we call the origin the point O, and after a rotational by an angle θ, the point where the circle touches the x-axis is T. Then the distance from O to T is

$$|OT| = \text{arc}PT = r\theta,$$

while the center of the circle is at $C = (r\theta, r)$.



Now draw a small right triangle, using the points P and C to define the hypotenuse, and a vertical line from C and a horizontal line from P forming the right angle. Call this point of the triangle Q. This will enable us to describe the coordinates of P in terms of x and y. Notice in the drawing that the angle at C of this triangle is θ. Thus the length of the bottom side is $|PQ| = r\sin\theta$, and the length of the vertical side is $|QC| = r\cos\theta$. The x coordinate of P then is just the distance the circle has traveled minus the length of the triangle bottom. And the y coordinate of P is then simply the height of the circle minus the length of the vertical side of the triangle. Thus we get

$$\begin{aligned} x = x(\theta) &= |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta), \text{ and} \\ y = y(\theta) &= |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta). \end{aligned}$$

Now suppose $x(t) = f(t)$ and $y(t) = g(t)$ is a parameterization, where both $f(t)$ and $g(t)$ are differentiable function of t. The derivatives actually measures how fast the respective coordinates x and y are changing as we vary t. As in the schematic diagram below, where the curve $x^2 + y^2 = 1$ is parameterized by $x(t) = \cos t$ and $y(t) = \sin t$, as t increases through the values of $t = 0$ and $t = \frac{\pi}{2}$, we can watch just how $\frac{dx}{dt}$ and $\frac{dy}{dt}$ change. As t passes through $t = 0$, we see the x-coordinate rise, peak, and then fall. Thus $\frac{dx}{dt}$ should be positive, hit a zero at $t = 0$ and then fall afterwards. Indeed, via calculation, $\frac{dx}{dt} = x'(t) = -\sin t$, which evaluates to 0 at $t = 0$, and was positive before it and negative after it. Similarly, $\frac{dy}{dt}$ should be positive throughout the pass through $t = 0$, and indeed, $\frac{dy}{dt} = y'(t) = \cos t > 0$ everywhere near $t = 0$. We can play the same game near the point $t = \frac{\pi}{2}$, and get analogous results. Work this out.

Now suppose that you have an equation involving y and y that expresses y implicitly as a function of x. Then even if you cannot solve for y, recall that you can still calculate the derivative of the implicit function, $\frac{dy}{dx}$ (this is implicit differentiation, which you played with in Calculus I).

Example 8. Let $x^2 + y^2 = 1$. Then

$$\begin{aligned} \frac{d}{dx}\left[x^2 + y^2\right. &= \left.1\right] \\ 2x + 2y\frac{dy}{dx} &= 0. \end{aligned}$$

which implies that $\frac{dy}{dx} = -\frac{x}{y}$. Consider the point on the curve $(x_0, y_0) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. You should show that this point is actually ON the curve. Here, with our implicit derivative,

$$\frac{dy}{dx}\bigg|_{(x_0,y_0)} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}.$$

The question arises, can we appeal directly to the parameterization, and calculate the slope of the line tangent to the curve at the point directly using the time derivatives of the coordinate functions in the parameterization? of course the answer is yes.

To see how, suppose for the moment that one can solve the implicit function for y, so $y = F(x)$. we assume also that it is differentiable, so that $\frac{dy}{dx} = F'(x)$, and the slope of the line tangent to the graph of the solutions of the equation $y = F(x)$ at the point (x_0, y_0), is $\frac{dy}{dx}$ evaluated at that point. using the parameterization, we get the composition $y(t) = F(x(t))$, which when we differentiate with respect to t (this time), we get

$$y'(t) = F'(x(t))x'(t) = F'(x)x'(t).$$

We can now immediately solve for $f'(x)$ and get

$$F'(x) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

This leads to the following conclusion:

Proposition 9. *Given a curve in the plane along with a differentiable parameterization $x(t)$ and $y(t)$, the slope of the line tangent to the graph of the curve at $t = t_0$ is*

$$\frac{dy}{dx}\bigg|_{(x(t_0),y(t_0))} = \frac{\frac{dy}{dt}\Big|_{t=t_0}}{\frac{dx}{dt}\Big|_{t=t_0}},$$

as long as $\frac{dx}{dt}\Big|_{t=t_0} \neq 0$.

So what happens when $\frac{dx}{dt}\Big|_{t=t_0} = 0$? go back to the original circle parameterization when we started discussing the derivatives of the parameterization. When $t = 0$ of the parameterization $x(t) = \cos t$ and $y(t) = \sin t$, we have that $\frac{dx}{dt}\Big|_{t=0} = 0$, and $\frac{dy}{dt}\Big|_{t=0} = 1$. The tangent line slope is not defined. Really, there is a tangent line here. It just has a vertical slope (the tangent line has equation $x = 1$).

So what happens when both $\frac{dx}{dt}\Big|_{t=t_0} = 0$ and $\frac{dy}{dt}\Big|_{t=t_0} = 0$? This time, go back to the parameterization $x(t) = \sin t$, and $y(t) = \sin^2 t$, and evaluate this at the point $t = \frac{\pi}{2}$. This is the point $(1, 1)$ in the plane and lies right at the edge of the curve. Here,

$$\frac{dx}{dt}\bigg|_{t=\frac{\pi}{2}} = \cos\frac{\pi}{2} = 0, \text{ and } \frac{dy}{dt}\bigg|_{t=\frac{\pi}{2}} = 2\sin\frac{\pi}{2}\cos\frac{\pi}{2} = 0.$$

The tangent line is again not defined. But this time, the curve actually stops at this point, so the tangent cannot be calculated based on a limit from one side. But also, the parameterization momentarily stops at this point, before doubling back on itself in the other direction. This is another way that tangent line may not exist on parameterized curves.

Note, however, in both of these case, the actual parameterizations were differentiable (each of $x(t) = f(t)$ and $y(t) = g(t)$ had derivatives everywhere). However, the PATH of the parameterization, or the curve had problems computing the tangent line. Corners and/or point where the curve stops are common in parameterized curves. They are non-differentiable points of the curve, even if they may be differentiable points of the parameterization. Strange, eh?

Lecture 3

One more example:

Example 10. Let $x(t) = t^2$, and $y(t) = t^3 - 3t$. Compute the equation of the line tangent to the curve at the point $(0, 0)$? At $(3, 0)$?

Can you see the problems with these points? First, let's eliminate the parameter t to create the equation that defines y implicitly as a function of x: It does not look straightforward, but we cannot solve either parameter function for the t variable. But if we square $y(t)$, then we can substitute in the $x(t)$ into the result:

$$y^2(t) = \left(t^3 - 3t\right)^2 = t^6 - 6t^4 + 9t^2 = x^3 - 6x^2 + 9x = x(x-3)^2.$$

Now we can calculate $\frac{dy}{dx}$ implicitly:

$$\begin{aligned} 2y\frac{dy}{dx} &= 3x^2 - 12x + 9 = 3(x-3)(x-1) \\ \frac{dy}{dx} &= \frac{3(x-3)(x-1)}{2y}. \end{aligned}$$

Now try to substitute in $t = 0$, getting $x(0) = 0$, and $y(0) = 0$. The derivative is not defined, with the problem being that the numerator is a non-zero number at $t = 0$, while the denominator is 0. Really this is not problematic at all, and actually denotes a vertical tangent line here (because both the numerator and the denominator are non-zero "near" the $t = 0$ point, the limit as this "rational" function goes to infinity as we approach $t = 0$). However, trying to substitute in the point $x = 3$, $y = 0$ yields a different kind of "does not exist" phenomenon. This point is more problematic, since the actual curve crosses itself at this point, and does so at different angles. The fact that the implicitly defined derivative doesn't exist for the curve is an indication that something is amiss here. Let's go to the parameterization to figure out why. Given the parameterization, since the y coordinate is 0, we find that there are two values of t that satisfy the equation $0 = y(t) = t^3 - 3t$, while also satisfying $3 = x(t) = t^2$. Namely, $t = \sqrt{3}$ and $t = -\sqrt{3}$. Since these are the only two solutions to these two equations, and since both $y(t)$ and $x(t)$ are differentiable functions of t, the curve must cross itself here. Choose a value of t and see if one can calculate the slope of the tangent line using the parameterization: First,

$$\begin{aligned} x'(t) &= \left.\frac{dx}{dt}\right|_{t=\sqrt{3}} = \left.2t\right|_{t=\sqrt{3}} = 2\sqrt{3} \\ y'(t) &= \left.\frac{dy}{dt}\right|_{t=\sqrt{3}} = \left.(3t^2 - 3)\right|_{t=\sqrt{3}} = 6. \end{aligned}$$

So

$$\left.\frac{dy}{dx}\right|_{(x(\sqrt{3}), y(\sqrt{3}))} = \frac{\left.\frac{dy}{dt}\right|_{t=\sqrt{3}}}{\left.\frac{dx}{dt}\right|_{t=\sqrt{3}}} = \frac{6}{2\sqrt{3}} = \frac{3}{sqrt3} = \sqrt{3} > 0,$$

and the equation of the line tangent to the curve is

$$y - 0 = \sqrt{3}(x - 3), \text{ or } y = \sqrt{3}(x - 3).$$

This should make sense, as the curve is rising from below the x-axis to above it as we pass through the point $t=\sqrt{3}$, in a way that $x(t)$ is also increasing (see the figure).

As for the other value of t corresponding to $(x,y)=(3,0)$, namely $t=-\sqrt{3}$, we get a tangent line equation of $y=-\sqrt{3}(x-3)$. This also makes sense, as like in the last case, the curve is rising as we increase the t-value through $t=-\sqrt{3}$, but in this case, the $x(t)$ function, though positive, is decreasing here (meaning that $x'(t)<0$ near $t=-\sqrt{3}$.

Here is another idea: if a curve is parameterized by t, this is like putting a ruler on the curve (we can measure distance along the curve using the parameter t. What would be the length of a curve given a parameterization and how would that compare to the length of the curve using the implicitly defined equation?

From Calculus I, we have: Given a curve $y=F(x)$, for $a\leq x\leq b$, the length of the curve given by the graph of $F(x)$ on the interval $[a,b]$ is given by

$$L=\int_a^b\sqrt{1+\left(F'(x)\right)^2}\,dx=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

using the parameterization, we can calculate this quantity using the parameter t, as long as we are careful to be consistent with this technique.

Recall the Substitution Rule: For a substitution $x=s(t)$, with $dx=s'(t)\,dt$, the Substitution Method yields

$$\int_\alpha^\beta\sqrt{1+\left(F'\left(s(t)\right)\right)^2}\,s'(t)\,dt=\int_{x(\alpha)}^{x(\beta)}\sqrt{1+\left(F'(x)\right)}\,dx,$$

where $x(\alpha)=a$ and $x(\beta)=b$. In this way, on the right we are integrating with respect to x along the curve. On the left, we are integrating direcly on the curve with respect to t. We can manipulate the expression on the left, where really the inside function is $x(t)$, to make it more palatable:

$$\begin{aligned}
L=\int_\alpha^\beta\sqrt{1+\left(F'\left(x(t)\right)\right)^2}\,x'(t)\,dt&=\int_\alpha^\beta\sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,\frac{dx}{dt}\,dt\\
&=\int_\alpha^\beta\sqrt{\left(\frac{\frac{dx}{dt}}{\frac{dx}{dt}}\right)^2+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,\frac{dx}{dt}\,dt\\
&=\int_\alpha^\beta\frac{1}{\frac{dx}{dt}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,\frac{dx}{dt}\,dt=\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.
\end{aligned}$$

Note that this all makes sense due to one of our previous calculations, namely that $y'(t)=\frac{d}{dt}\left[F(x(t))\right]=F'(x(t))x'(t)$, so that $F'(x(t))=\frac{y'(t)}{x'(t)}$.

Thus the length of a curve parameterized by t on the interval from $t=\alpha$ to $t=\beta$ is given by this last definite integral.

Example 11. What is the circumference C of the unit circle? From years back in a geometry class, we know it as π times the diameter of the circle, or $2\pi r$, where r is the radius. Hence C, for a circle of radius 1, is 2π. Given a parameterization, say the standard one $x(t)=\cos t$ and $y(t)=\sin t$, the interval $[0,2\pi]$ in

t traverses the circle exactly once (this part is important). Hence the length of the perimeter of the circle is

$$C = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{1}\, dt = \int_0^{2\pi} dt = t\Big|_0^{2\pi} = 2\pi.$$

Exercise 1. Now do the same calculation with the parameterization $x(t) = \sin 4\pi t$ and $y(t) = -\cos 4\pi t$. The interval limits will change here, but the final result will not.

Another idea? What if we needed to find the area between a curve (say it is the curve given by a positive function $y = F(x)$ on an interval $[a, b]$), and the x-axis, and we ONLY had the parameterization of the curve? Can we still use the parameterization to find this area? Of course the answer is yes. The parameterization offers exactly the same information as that of the possibly unknown function $y = F(x)$. Hence determining how to do this calculation again only involves the proper translating of the information from what you know to what you have yet to find out.

In this case, given $y = F(x)$, where just for the sake of the argument, assume $F(x) > 0$ on an interval $[a, b]$, the area between the graph of $F(x)$ and the x-axis is

$$A = \int_a^b F(x)\, dx.$$

Given the parameterization $x(t)$ and $y(t)$, the point $a = x(\alpha)$ for some $t = \alpha$, and $b = x(\beta)$, for some $t = \beta$. Then by the same type of argument as above,

$$A = \int_a^b F(x)\, dx = \int_{x(\alpha)}^{x(\beta)} F(x)\, dx = \int_\alpha^\beta F(x(t))x'(t)\, dt = \int_\alpha^\beta y(t)x'(t)\, dt,$$

since $y(t) = F(x(t))$. Really, that is it and is again just a reinterpretation (backwards in a way) of the Anti-Chain rule, or the Substitution Method.

We will continue next time with an example.