110.109 CALCULUS II

Week 12 Lecture Notes: April 23 - April 27

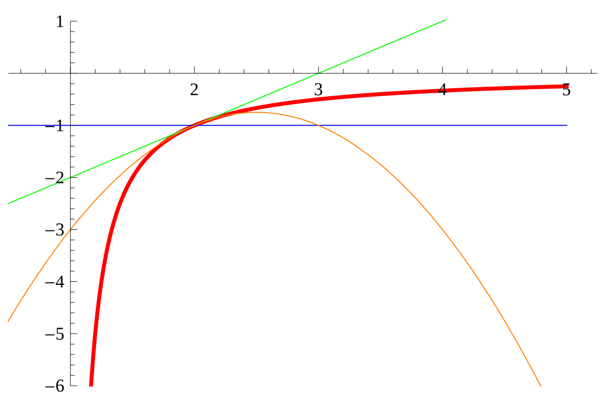
Lecture 2: Section 10.10 Taylor and Maclauren series

Let's go back to our geometric series, $g(x) = \dfrac{1}{1-x}$, expanded at $a = 2$:

$$g(x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n = -1 + (x-2) - (x-2)^2 + (x-2)^3 - (x-2)^4 + \cdots \text{ for } -1 < x < 1.$$

This time, call $T_n(x)$ the nth Taylor polynomial of $g(x)$, at $x = 2$. Here

$$\begin{aligned}
T_0(x) &= -1 \\
T_1(x) &= -1 + (x-2) = x - 3 \\
T_2(x) &= -1 + (x-2) - (x-2)^2 = -x^2 + 5x - 7 \\
T_3(x) &= -1 + (x-2) - (x-2)^2 + (x-2)^3 \\
&\vdots \\
T_n(x) &= \sum_{i=0}^{n} (-1)^{i+1}(x-2)^i.
\end{aligned}$$



These polynomials are considered the "best" degree-n polynomials to approximate $g(x)$ at and near the point $x = 2$. The main reason is because they are the polynomials that have the same derivatives (up to order-n for $T_n(x)$ as that of $g(x)$. This is by design, and the Taylor series is constructed using the derivatives. As an example, it is a fact from Calculus I that the tangent line is the best linear approximation to a function at a point (if it exists, that is). Here, notice that $g(2) = -1$, and $g'(2) = \frac{d}{dx}\left[\frac{1}{1-x}\right]\Big|_{x=2} = \left[\frac{1}{(1-x)^2}\right]\Big|_{x=2} = 1$. Thus

$$T_1(x) = -1 + (x-2) = g(2) + g'(2)(x-2),$$

IS the equation of the tangent line and thus is the best linear function to approximate $g(x)$ at $x - 2$. The best quadratic will be $T_2(x)$ since it has the same 0th, 1st and 2nd derivatives as that of $g(x)$ at $x - 2$. Note the graph below.

Date: April 27, 2012.

Given that each $T_n(x)$ is just the partial sum of the series (and where the series converges, we have the series equals $g(x)$), we get that $\lim_{n\to\infty} T_n(x) = g(x)$ (on the interval of convergence). Keep in mind here that the partial sums, as well as the series still include the unspecified variable x. Hence each is really still a function of x.

How one shows that the limit of the Taylor polynomials (as n goes to ∞) is the function also gives a way of estimating just hoe good an approximation to $g(x)$ the nth Taylor polynomial is: Let

$$R_n(x) = T_n(x) - g(x)$$

be the *remainder* of the nth Taylor polynomial. It is again a function of x (since it is the difference of two functions of x). Note that if $\lim_{n\to\infty} T_n(x) = g(x)$, then

$$\lim_{n\to\infty} T_n(x) - g(x) = 0 = \lim_{n\to\infty} R_n(x).$$

Showing this may be difficult. However, we have a good way to estimate just hoe big the remainder can be.

Theorem 1. *If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| < d$, then the remainder $R_n(x)$ of the nth Taylor polynomial satisfies the inequality*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}, \text{ for } |x-a| < d.$$

Some notes:

- In this theorem, they use d to denote the radius of convergence of the series. This is to avoid the confusion of using R for both the radius of convergence and the remainder.
- Really, the right-hand-side of the inequality in the theorem is just a cap on the size on the NEXT term in the series after the nth Taylor polynomial. To compare directly, notice

$$\underbrace{\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}}_{(n+1)\text{st term in Taylor series}} \qquad \underbrace{\frac{M}{(n+1)!}|x-a|^{n+1}}_{\text{bound for } R_n(x)}.$$

  You can see directly where the bound $\left|f^{(n+1)}(x)\right| \leq M$ comes in, as well as the switch from the parentheses in $(x-a)^{n+1}$ to the absolute values $|x-a|^{n+1}$.

Notice that there are many good examples of Taylor series in this section. It will pay well to spend some time with these, and not just skim over them. Here is one of them.

Example 2. *Let $k \in \mathbb{N}$ be a natural number. Expand $f(x) = (1+x)^k$ as a Maclauren series.* As before, we start by finding either a pattern for the derivatives of $f(x)$ at $x = 0$, or at least calculating them. Note first here, though, that for any k, $f(x)$ IS a polynomial. So $f(x)$ has derivatives of all orders, even though after k, they will all be 0. Here

$$\begin{aligned}
f(0) &= (1+x)^k\Big|_{x=0} = 1\\
f'(0) &= k(1+x)^{k-1}\Big|_{x=0} = k\\
f''(0) &= k(k-1)(1+x)^{k-2}\Big|_{x=0} = k(k-1)\\
f^{(3)}(0) &= k(k-1)(k-2)(1+x)^{k-3}\Big|_{x=0} = k(k-1)(k-2)\\
\vdots \quad & \quad \vdots\\
f^{(n)}(0) &= k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n} = k(k-1)(k-2)\cdots(k-n+1) = \frac{k!}{(k-n)!},
\end{aligned}$$

and thus all derivatives are 0 after the nth (can you see this?). Thus

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \left(\frac{k!}{(k-n)!}\right)\frac{1}{n!}x^n = \sum_{n=0}^{k}\left(\frac{k!}{(k-n)!n!}\right)x^n = \sum_{n=0}^{k}\binom{k}{n}x^n,$$

where the notation $\binom{k}{n}$ is the standard notation for the counting principle of how many ways one can choose n objects out of a set of k objects where the order of the choosing does not matter. Thus the coefficients of the polynomial $f(x) = (1+x)^k$ are the entries in the kth row of Pascal's Triangle

$$\begin{array}{lccccccccccc}
k=0: & & & & & & 1 & & & & & \\
k=1: & & & & & 1 & & 1 & & & & \\
k=2: & & & & 1 & & 2 & & 1 & & & \\
k=3: & & & 1 & & 3 & & 3 & & 1 & & \\
k=4: & & 1 & & 4 & & 6 & & 4 & & 1 & \\
k=5: & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

and so

$$f(x) = (1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

Lecture 2: Section 10.10 Taylor and Maclauren series (cont'd.)

One thing about the binomial series mentioned above is that, for $k \in \mathbb{R}$ but $k \notin \mathbb{N}$, then the function $f(x) = (1+x)^k$ is not a polynomial. But it still has derivatives that behave roughly the same as when k is a natural number. It is just that in general, the derivatives will not wind up being 0 after a while. Thus the Maclauren series of this $f(x)$ should still exist, but will not be a finite series. And the coefficients should wind up looking a lot like those above for k a natural number. All of this is true, and can be written down explicitly through the idea of *generalized binomial coefficients*: For $k, n \in \mathbb{N}$, we know

$$\binom{k}{n} = \frac{k!}{(k-n)!n!} = \left(\frac{k!}{(k-n)!}\right)\frac{1}{n!} = \frac{\left(\frac{k!}{(k-n)!}\right)}{n!},$$

where the numerator is the thing to focus on here. Indeed,

$$\begin{aligned}\frac{k!}{(k-n)!} &= \frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)(k-n-1)\cdots 1}{(k-n)(k-n-1)\cdots 1} \\ &= \frac{k(k-1)(k-2)\cdots(k-n+1)\cdot(k-n)!}{(k-n)!} \\ &= \overbrace{k(k-1)(k-2)\cdots(k-n+1)}^{n \text{ terms}} = \prod_{i=0}^{n-1}(k-i).\end{aligned}$$

The n-terms on the right are simply the product of k and each of its n predecessors (a predecessor here is defined as the number formed by decrementing k by 1). But this will also work if $k \notin \mathbb{N}$ is any real number, as in

$$\binom{\pi}{2} = \frac{\left(\prod_{i=0}^{2-1=1}(\pi-i)\right)}{2!} = \frac{\pi(\pi-1)}{2} \quad \text{or}$$

$$\binom{\pi}{5} = \frac{\left(\prod_{i=0}^{4}(\pi-i)\right)}{5!} = \frac{\pi(\pi-1)(\pi-2)(\pi-3)(\pi-4)}{5!}.$$

The only difference between this new case and the regular version of binomial coefficients you are familiar with (as polynomial coefficients), is that now, the bottom number n may be larger than the top number k. But this was true even in the regular case, as

$$\binom{5}{2} = \frac{5!}{(5-2)!2!} = \frac{\left(\prod_{i=0}^{2-1=1}(5-i)\right)}{2!} = \frac{5(5-1)}{2} = 10, \qquad \text{while}$$

$$\binom{5}{7} = \frac{\left(\prod_{i=0}^{7-1=6}(5-i)\right)}{7!} = \frac{5(5-1)(5-2)(5-3)(5-4)(5-5)(5-6)}{7!} = 0.$$

But when k is not a natural number, the generalized binomial coefficients do not become 0 after a while (in general).

Some Notes:

- If $k \notin \mathbb{N}$, then $(1+x)^k$ can still be written as a power series as
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n,$$
except that now the series is NOT finite (the resulting function is NOT a polynomial).
- It is a bit tricky to show (it is in the book, however), but the radius of convergence for this power series in $R = 1$.

Example 3. *Find the Maclauren series for* $i(x) = \dfrac{1}{\sqrt{1+x}}$*.* Here we could simply start calculating the derivatives of $i(x)$, setting them to 0, hopefully find a pattern and then write the series using the pattern. However, we can also recognize that this function can be written as an (infinite) binomial series,

$$i(x) = (1+x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n,$$

where $k = -\frac{1}{2}$. This series looks like

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n &= \binom{-\frac{1}{2}}{0} x^0 + \binom{-\frac{1}{2}}{1} x^1 + \binom{-\frac{1}{2}}{2} x^2 + \binom{-\frac{1}{2}}{3} x^3 + \binom{-\frac{1}{2}}{4} x^4 + \dots \\ &= 1 + \frac{\left(-\frac{1}{2}\right)}{1!} x^1 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} x^3 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!} x^4 + \dots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{15}{48}x^3 + \frac{105}{384}x^4 + \dots.\end{aligned}$$

Example 4. *Use the fourth Taylor polynomial* $T_4(x)$ *of* $i(x) = \dfrac{1}{\sqrt{1+x}}$ *to estimate* $\sqrt{2}$*.* Here, we note two things: First, we have

$$i\left(-\frac{1}{2}\right) = \frac{1}{\sqrt{1+\left(-\frac{1}{2}\right)}} = \frac{1}{\sqrt{\frac{1}{2}}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2},$$

and second, since the radius of convergence for the binomial series is $R = 1$, and $-\frac{1}{2} \in (-1, 1)$, the function equals the power series at this value, and we can use the series to estimate the function value. From above,

$$i(x) \cong T_4(x) = \sum_{n=0}^{4} \binom{-\frac{1}{2}}{n} x^n = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{15}{48}x^3 + \frac{105}{384}x^4,$$

and evaluated at $x = -\frac{1}{2}$, we get

$$\begin{aligned} T_4\left(-\frac{1}{2}\right) &= 1 - \frac{1}{2}\left(-\frac{1}{2}\right) + \frac{3}{8}\left(-\frac{1}{2}\right)^2 - \frac{15}{48}\left(-\frac{1}{2}\right)^3 + \frac{105}{384}\left(-\frac{1}{2}\right)^4 \\ &\cong 1.38. \end{aligned}$$

Incidentally, a better approximation of the square root of 2 is $\sqrt{2} \cong 1.4142136$.

Taylor series can be quite useful for things like integration. Some functions are quite difficult, or impossible to find antiderivatives for (at least in a nice form). One such function is e^{-x^2}, an expression intimately related to the standard normal curve in statistics. How so? The Gaussian Distribution is a continuous probability distribution given by the function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where μ is the mean of the distribution, and σ is the standard deviation. Take a distribution with $\mu = 0$ and $\sigma = \frac{1}{\sqrt{2}}$, and you get

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}.$$

Finding the area under $f(x)$ amounts to finding the probability that the value of a normal random variable takes the value of x or less, or

$$P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx.$$

Hence being able to anti-differentiate e^{-x^2} would be very helpful. However, there is no nice expression for a function whose derivative is $f(x)$.

Example 5. *Calculate* $\int_0^1 e^{-x^2}\, dx$ *via a power series approximation to within .001.* First, we seek to write the integrand as a power series. Here

$$e^{-x^2} = e^{(-x^2)} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}.$$

One can use the Ratio Test here to ensure that this power series equals the function for all values of x (that is, the radius of convergence is$R = \infty$. You should do this on your own). As a power series, e^{-x^2} is easy to integrate, and the antiderivative of e^{-x^2} is

$$\int e^{-x^2}\, dx = \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \right) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}.$$

This new series will again converge for any choice of x (why is this the case?). Hence, this series equals the antiderivative of e^{-x^2} on the interval $[0,1]$. And so

$$\int_0^1 e^{-x^2}\,dx = \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{n!(2n+1)}\right)\Big|_0^1 = \sum_{n=0}^{\infty}(-1)^n \frac{1^{2n+1}}{n!(2n+1)} - \sum_{n=0}^{\infty}(-1)^n \frac{0^{2n+1}}{n!(2n+1)}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \ldots .$$

Now this last series is an alternating series. Recall that for an alternating series $\sum_{n=0}^{\infty}(-1)^n b_n$ which converges, the nth partial sum s_n, is within b_{n+1} of the true sum (whatever that is). Hence we have $|s_n - s| < b_{n+1}$. In our case, notice that

$$b_5 = \frac{1}{1320} < \frac{1}{1000} = .001,$$

and hence we know that the partial sum

$$s_4 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \cong .74749$$

is within .001 of the true sum. Incidentally, a computer generated better approximation for the true sum is something like .7468.

The Taylor series of a product of functions can be calculated in the normal direct way (by taking derivatives, evaluating them at a point, and then looking for a pattern), or by simply writing the power series for each of the product functions and then multiplying the two power series (term by term, that is). Recall when multiplying two polynomials, each term in one polynomial must be multiplied to each term in the other . While this may be tedious and time consuming, remember that the early terms in a power series are the most important, and there are very few calculations needed to determine these early terms. To see this, note the following example:

Example 6. *Determine the Maclauren series for $e^x \sin x$.* Taking multiple derivatives of this function may involve many terms to juggle. Instead, let's simply multiply the power series of each factor function together: We get

$$\begin{aligned}
e^x \sin x &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\dots\right) \\
&= 1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\dots\right) + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\dots\right) \\
&\qquad + \frac{x^2}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\dots\right) + \frac{x^3}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\dots\right) + \dots \\
&= x + x^2 + \left(-\frac{1}{3!} + \frac{1}{2!}\right)x^3 + \left(-\frac{1}{3!} + \frac{1}{3!}\right)x^4 + \left(-\frac{1}{5!} + \frac{1}{2!3!} + \frac{1}{4!}\right)x^5 + \dots \\
&= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots .
\end{aligned}$$

This also works for function that can be written as the quotient of two other functions.

Example 7. *Find the Maclauren series of* $\tan x$*.* Again, the direct way would be to calculate the derivatives of $\tan x$ and set them all to 0. But This starts to get complicated as one winds up using the product rule a lot after the first couple of derivatives. Not that this is any easier, but you can also do the following:

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)}{\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right)}.$$

To write out the series, one would have to use long division and calculate

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\dots\right)\overline{\Big)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\dots\right)}.$$

While this seems difficult, it is doable.

Next class, we will run through an example of calculating the power series of a product of functions using the direct method as well as the product method.