110.109 CALCULUS II

Week 11 Lecture Notes: April 16 - April 20

Lecture 1: Power Series

We start today with two more examples. But first, recall from the last lecture that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ converged by the Ratio (or the Root) test as long as we chose values for x that satisfied the inequality $|x-5| < 1$. This describes the interval $4 < x < 6$. Contrast this to the following examples:

Example 1. *Test the series* $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$ *for convergence.* Let's again appeal to the Ratio Test here, with $a_n = \frac{(x-1)^n}{n!}$ and $a_{n+1} = \frac{(x-1)^{n+1}}{(n+1)!}$. We see

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-1)^{n+1}}{(n+1)!}}{\frac{(x-1)^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \right| \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} |x-1| \cdot \frac{1}{n+1} = |x-1| \lim_{n\to\infty} \frac{1}{n+1} = |x-1| \cdot 0 = 0$$

regardless of our choice of x. In fact, for any choice of x, the series will converge.

Example 2. *Test the series* $\displaystyle\sum_{n=1}^{\infty} n^n (x-3)^n$ *for convergence.* For this series, we first make two observations:

- The series will certainly converge when $x = 3$, since then all terms are 0. In fact, ANY power series centered at a will converge when the expression $(x-a) = 0$, or when $x = a$. Thus there will always be at least one point where any power series will converge.
- Notice the coefficients here: $c_n = n^n$. They will be growing quite fast as n increases. Start building your intuition and try to guess for what values of x this series will converge.

Here again, we have a choice of tests. I will choose the Root Test here, but it won't matter. Noticing that the series will converge for $x = 3$, let's use the Root Test under the assumption that we are testing values of x OTHER than 3: For $x \neq 3$, we see that

$$\lim_{n\to\infty} = \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|n^n (x-3)^n|}$$

$$= \lim_{n\to\infty} \sqrt[n]{n^n |x-3|^n} = \lim_{n\to\infty} n|x-3| = |x-3| \lim_{n\to\infty} n = \infty$$

Date: April 18, 2012.

for any choice of x not equal to 3. Hence there are NO OTHER values of x where the series converges.

So in the last three examples, we have seen that sometimes a series centered at a will converge for only one value of x (namely $x = a$), a finite interval centered at a, or for ALL values of x. It turns out, these three types of conclusions are the ONLY valid ones for series:

Theorem 3. *Given a power series centered at a point $x = a$, $\sum c_n(x-a)^n$, either*

(1) the series converges only when $x = a$,
(2) the series converges for all $x \in \mathbb{R}$, or
(3) there is a positive number $R > 0$ where the series converges when $|x - a| < R$ and diverges when $|x - a| > R$.

Some notes:

- In the third case, the condition $|x - a| < R$ for convergence describes a finite-length interval centered at $x = a$, and extends out in both directions R units away. The interval is $-R < x - a < R$, or $a - R < x < a + R$.
- The number R here is called the *radius of convergence*. It is called a radius since the interval, centered at a, extends out by R in both directions. One can generalize the theorem to use R is all three cases: either $R = 0$ in the first case, or $R = \infty$ in the second, or $0 < R < \infty$ in the third.
- At the edges of the interval in the third case, the series may in fact converge when $x = a - R$, or when $x = a + R$. The Ratio or Root Tests would be inconclusive at these precise points. In fact, each of these two values for x will have to be tested separately for convergence. In case three. the ultimate set of all values of x for which the series converges is called the *interval of convergence* and may take any of the forms

$$[a-R, a+R]\,, \quad (a-R, a+R]\,, \quad [a-R, a+R)\,, \quad \text{or} \quad (a-R, a+R)\,.$$

Some more examples:

Example 4. *Back to the example $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$. Find the interval of convergence.* We found here that the radius of convergence was $R = 1$, since we found that the series converged when $|x - 5| < 1$ by the Ration Test (or the Root Test). At the end points?

Let $x = 4$: Then the series becomes

$$\sum_{n=1}^{\infty} \frac{(4-5)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which is the Alternating Harmonic Series. We know this converges from our past analysis, and hence the value $x = 4$ is part of the interval of convergence.

Now let $x = 6$: Here the series becomes

$$\sum_{n=1}^{\infty} \frac{(6-5)^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is just the Harmonic Series. As the Harmonic Series diverges, the value $x = 6$ is NOT part of the interval of convergence.

Hence the interval of convergence is $4 \le x < 6$, or the interval $[4, 6)$.

Note that the examples in the text are quite good. Follow them closely!

Example 5. *Find the interval of convergence for the series* $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[n]{n}}$. We will employ the Ratio Test here again. With $a_n = \frac{(-2)^n x^n}{\sqrt[n]{n}}$ and $a_{n+1} = \frac{(-2)^{n+1} x^{n+1}}{\sqrt[n+1]{n+1}}$, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1} x^{n+1}}{\sqrt[n+1]{n+1}}}{\frac{(-2)^n x^n}{\sqrt[n]{n}}} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1} x^{n+1}}{\sqrt[n+1]{n+1}} \cdot \frac{\sqrt[n]{n}}{(-2)^n x^n} \right|$$

$$= \lim_{n\to\infty} |-2x| \cdot \frac{\sqrt[n]{n}}{\sqrt[n+1]{n+1}} = |-2x| \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n+1]{n+1}}.$$

This last limit is slightly different from what we have encountered previously. We cannot simply move the limit into the inside of the radical, since the radical is a function of n. Hence we need to be a bit more clever. Recall that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. But the sequence $\left\{ n^{\frac{1}{n}} \right\}$ is exactly the same as the sequence $\left\{ (n+1)^{\frac{1}{n+1}} \right\}$ except that one is moved over by one term. Hence their limits at infinity are also precisely the same (does this make sense to you?). Hence it is also the case that $\lim_{n\to\infty} (n+1)^{\frac{1}{n+1}} = 1$. This is crucial, since then

$$|-2x| \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n+1]{n+1}} = |-2x| \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{(n+1)^{\frac{1}{n+1}}} = 2|x| \frac{\lim_{n\to\infty} n^{\frac{1}{n}}}{\lim_{n\to\infty} (n+1)^{\frac{1}{n+1}}} = 2|x| \frac{1}{1} = 2|x|,$$

since by the quotient rule for limits, the limit of a quotient IS the quotient of the limits as long as each of the individual limits exist and the limit of the denominator is not 0. Now, since the Ratio Test says that the series will converge as long as the above limit is less than one, we get the criteria

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1} x^{n+1}}{\sqrt[n+1]{n+1}}}{\frac{(-2)^n x^n}{\sqrt[n]{n}}} \right| = 2|x| < 1.$$

If we put this in the form given by the previous theorem, we get $|x| = |x = 0| < \frac{1}{2}$. Hence for this series, centered at 0, the radius of convergence is $R = \frac{1}{2}$. To test the endpoints:

Let $x = -\frac{1}{2}$. Then $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n \left(-\frac{1}{2}\right)^n}{\sqrt[n]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$. Here, by the Divergence Test, this series diverges, since the terms $a_n = \frac{1}{\sqrt[n]{n}}$ do not go to 0, as n goes to infinity. Hence $x = -\frac{1}{2}$ is NOT in the interval. By the same argument, neither is $x = \frac{1}{2}$, and the interval of convergence is precisely $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Lecture 2: Section 10.9 Representing Functions as Power Series

Let's go back to our geometric series, written as a function where the series converges:

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \text{ for } -1 < x < 1.$$

This idea is fairly profound, as it equates a function like the rational function on the right with an infinite degree polynomial. Keep in mind, though, that the domain of the function $\frac{1}{1-x}$ is all $x \neq 1$, and the series only converges for $x \in (-1, 1)$. Hence, this series equals the function only on the domain where the series actually makes sense (read: converges).

Other functions that can be manipulated into looking like the one above can now also be written as power series. For some examples:

Example 6. Let $g(x) = \frac{1}{1-x^2}$. Here we can see directly that $g(x) = f(x^2)$, and $g(x)$ is a composite function of the above $f(x)$ with x^2. Hence we can write

$$g(x) = \frac{1}{1-x^2} = \frac{1}{1-(x^2)} = f(x^2) = \sum_{n=0}^{\infty} \left(x^2\right)^n = \sum_{n=0}^{\infty} x^{2n}.$$

Again, we would need to know where this series converges to know the domain for which we can equate the function $g(x)$ to the series. By the Root Test, where $a_n = x^{2n}$, we get that the series will converge where

$$\lim_{n\to\infty} \sqrt[n]{|x^{2n}|} = \lim_{n\to\infty} \sqrt[n]{\left(x^2\right)^n} = \lim_{n\to\infty} x^2 = x^2 \lim_{n\to\infty} 1 = x^2 < 1.$$

Hence the radius of convergence is $R = 1$. As for the endpoints, when $x = 1$ and when $x = -1$, the series does not converge (can you see this? Use the Divergence Test on these two series. You will find the terms do not go to 0). Hence we have that the interval of convergence is $(-1, 1)$, and

$$g(x) = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \ldots \text{ for } x \in (-1, 1).$$

Example 7. Let $h(x) = \frac{1}{1+x^2}$. Here again we can see that $h(x) = f(-x^2)$, and $h(x)$ is a composite function of the above $f(x)$ with $-x^2$. Hence we can write

$$h(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = f(-x^2) = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

As for where this alternating series converges, we appeal to the Ratio Test, where $a_n = (-1)^n x^{2n}$ and $a_{n+1} = (-1)^{n+1} x^{2(n+1)}$, we get that the series will converge where

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} x^2 = x^2 \lim_{n\to\infty} 1 = x^2 < 1.$$

Again, the radius of convergence is $R = 1$, and again when $x = 1$ and when $x = -1$, the series does not converge (you should work this out explicitly?). Hence the interval of convergence is $(-1, 1)$, and

$$h(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \dots \text{ for } x \in (-1, 1).$$

Example 8. Let $i(x) = \frac{1}{3-x}$. This time, we actually have two directions we can go in:

(1) Using a bit of algebraic manipulation, we can write

$$i(x) = \frac{1}{3-x} = \frac{1}{1-(x-2)} = f(x-2),$$

and $i(x)$ is a composite function of the above $f(x)$ with $x - 2$. Hence

$$i(x) = \frac{1}{1-(x-2)} = \sum_{n=0}^{\infty} (x-2)^n .$$

This is the standard geometric series centered on 2, instead of 0. As for where this alternating series converges, really, all of these are geometric, so one can almost guess where each of these series in these examples will converge. This one again will have a radius of convergence of $R = 1$, and the endpoints are not included. Careful here as the final result is that the interval of convergence is centered at 2, and

$$\begin{aligned} i(x) = \frac{1}{3-x} &= \sum_{n=0}^{\infty}(x-2)^n \\ &= 1 + (x-2) + (x-2)^2 + (x-2)^3 + \dots \text{ for } x \in (1, 3). \end{aligned}$$

(2) Another idea is to write

$$i(x) = \frac{1}{3-x} = \frac{1}{3-\frac{3x}{3}} = \frac{1}{3\left(1-\frac{x}{3}\right)} = \frac{1}{3}\left(\frac{1}{1-\frac{x}{3}}\right) = \frac{1}{3}f\left(\frac{x}{3}\right).$$

Then

$$i(x) = \frac{1}{3} f\left(\frac{x}{3}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{x^n}{3^n} = \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}.$$

So what is the interval of convergence here? First, we find the radius of convergence: Using the Ratio Test with $a_n = \frac{x^n}{3^{n+1}}$, and $a_{n+1} = \frac{x^{n+1}}{3^{n+2}}$, we get

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{3^{n+2}}}{\frac{x^n}{3^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|\lim_{n\to\infty} 1 = \left|\frac{x}{3}\right| < 1,$$

which is satisfies for $|x| < 3$. Hence $R = 3$ is the radius of convergence. Testing the endpoints ($x = \pm 3$ yields divergent series on each end. Hence the interval of convergence for this series is $-3 < x < 3$.

Either fashion would work here, and the two constructions are equally valid. The second way produces a power series which equals the function on a large domain. But regardless, both work.

Example 9. Let $j(x) = \frac{1}{3+x}$. This is exactly like the previous example with the single exception of replacing x with $-x$. We will only do one of the above methods here. Here

$$j(x) = \frac{1}{3+x} = \frac{1}{3\left(1+\frac{x}{3}\right)} = \frac{1}{3}\left(\frac{1}{1-\left(-\frac{x}{3}\right)}\right) = \frac{1}{3} f\left(-\frac{x}{3}\right).$$

For the series, then

$$j(x) = \frac{1}{3}\left(\frac{1}{1-\left(-\frac{x}{3}\right)}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{3^n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{3^{n+1}}.$$

The last little manipulation is not really necessary, but does clean things up a bit. Again, for the radius of convergence, we use the Ratio Test with $a_n = (-1)^n\frac{x^n}{3^{n+1}}$, and $a_{n+1} = (-1)^{n+1}\frac{x^{n+1}}{3^{n+2}}$ and get

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{x^{n+1}}{3^{n+2}}}{(-1)^n\frac{x^n}{3^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|\lim_{n\to\infty} 1 = \left|\frac{x}{3}\right| < 1,$$

which is satisfies for $|x| < 3$. Hence $R = 3$ is the radius of convergence. With the extra knowledge that the endpoints are not included (check this again), we have that the interval of convergence is $(-3, 3)$ and the final result is

$$j(x) = \frac{1}{3+x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{3^{n+1}} = \frac{1}{3} - \frac{x}{9} + \frac{x^2}{27} + \frac{x^3}{81} + \ldots \text{ for } x \in (-3, 3).$$

Note. *Power series' act just like infinite degree polynomials. Hence one can differentiate and integrate them term by term (the derivative/integral of a sum is the sum of the derivatives/integrals, no?), using nothing more than the Power/Anti-Power Rules. The result is* *another power series (the derivative of a polynomial IS a polynomial of one less degree. But if what you start with is an infinite degree polynomial, then the result after differentiation is another infinite degree polynomial (what about integration?)*

Theorem 10. *Let*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

be continuous and differentiable on the interval $(a-R, a+R)$. Then

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

and

$$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$$

The radius of convergence for each of these is equal to the original radius of convergence.

Remark 11. Notice that for the anti-derivative, we needed to introduce the constant of integration C, as usual. Notice also that the anti-derivative of the constant term c_0 is $c_0(x-a)$ instead of simply c_0x. This is done purely for symmetry. It seems like we added a new term $-ac_0$ to the mix. But with the constant of integration being an unknown, this extra term is just another constant and gets absorbed once one finds the value of C in a real problem. Plus, it makes the calculation as a series very efficiently specified, so this is why it is included this way.

Example 12. *Express $g(x) = \frac{1}{(1-x)^2}$ as a power series.* Here, this function does not look fit to be expressed as a geometric power series directly. However, with a little foresight, we can see that for $f(x) = \frac{1}{1-x}$, that

$$f'(x) = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2} = g(x).$$

This is very helpful, given the previous theorem, since $f(x)$ can be written as our standard power series, and we can differentiate it to get $g(x)$. So

$$g(x) = f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n.$$

Again, the last manipulation is not really needed. And again, the radius of convergence does not change here, so $g(x)$ equals this power series on the interval $(-1, 1)$.

Here are a couple of notes:

- The series corresponding to $g(x)$ above is NOT geometric, given the coefficients are a non-constant function of n. It was derived (literally) from a geometric series, though. Geometric series are very special, and somewhat rare.
- while the radius of convergence does NOT change under differentiation and integration, the interval of convergence may change. The endpoints will always need to be checked after differentiation or integration. Sorry about that.

Example 13. *Write $\ell(x) = \ln(1+x^2)$ as a power series.* Here, this function does not look remotely like a geometric series. However, it is related in a way. First, notice that

$$\ell'(x) = \frac{d}{dx}\left[\ln\left(1+x^2\right)\right] = \frac{2x}{1+x^2} = 2x\left(\frac{1}{1-(-x^2)}\right).$$

Written like this last term on the right, one can see that the derivative does indeed look like the term $2x$ multiplied by the function in Example 7 above. We already know how to write that function as a power series, and thus

$$\ell'(x) = 2x\left(\frac{1}{1-(-x^2)}\right) = 2x\left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right) = \sum_{n=0}^{\infty}2(-1)^n x^{2n+1}.$$

We can check immediately that the interval of convergence here is again $(-1,1)$ (which you should do).

This gives us the series representation for $\ell'(x)$. We want $\ell(x)$, which we can find via integration:

$$\ell(x) = \ln\left(1+x^2\right) = \int\left(\sum_{n=0}^{\infty}2(-1)^n x^{2n+1}\right)dx = C+\sum_{n=0}^{\infty}2(-1)^n\frac{x^{2n+2}}{2n+2} = C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{n+1},$$

where we found a way to kill off the 2 in the series terms (not necessary, actually). We need to find the correct value of C here, and since $\ell(0) = \ln(1+0^2) = 0$, we find that $C = 0$, and

$$\ln\left(1+x^2\right) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{n+1} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \ldots \text{ for } x \in (-1,1).$$

Note that Example 7, page 731 in the book is another good example to study.

Lecture 3