110.109 CALCULUS II

Week 10 Lecture Notes: April 9 - April 13

Lecture 3: The Root Test

There is one more test which in certain cases establishes that a series is absolutely convergent (and hence converges). This last test, like the Ratio Test previously, is one where one watches how the sequence of terms a_n in the series $\sum a_n$ goes to 0 (as it must if the series has any hope of converging!). Here, though, as n gets large and goes to ∞, we watch how the nth root of the term a_n changes. We do this since as the terms are going to 0 in magnitude, the nth root of a small number tends to grow back toward 1. Take a number less that 1, and take larger and larger powers of it. The resulting numbers will get smaller and smaller. Instead, if one takes smaller and smaller fractional powers of a number c between 0 and 1, that number will grow toward 1. Try this on the fraction $\frac{1}{2}$:

$$\left\{\left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{2^n}, \dots\right\} \longrightarrow 0, \quad \text{while}$$

$$\begin{aligned}\left\{\left(\frac{1}{2}\right)^{\frac{1}{n}}\right\} &= \left\{\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[4]{2}}, \dots, \frac{1}{\sqrt[n]{2}}, \dots\right\} \\ &= \{.5,\ .707107,\ .793701,\ .840896,\ .870551, \dots\} \longrightarrow 1.\end{aligned}$$

We have the following:

Theorem 1 (The Root Test)**.** *Given a series* $\sum a_n$*,*

(1) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$*, then* $\sum a_n$ *converges absolutely, and hence converges.*
(2) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$*, or the limit does not exist, then* $\sum a_n$ *diverges.*
(3) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$*, then the test is inconclusive.*

We won't prove the Root Test here, but we can say a lot of things about it:

- The absolute signs are necessary here so that the nth root makes sense for even integers n. Also, the number L, if it exists, must be non-negative, since each term in this particular sequence of roots of a_n will be positive.
- If $\sum a_n$ has any hope of converging, then $\lim_{n\to\infty} a_n = 0$. But if the second par of the Root Test is true, then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then after a while (for large values of n), it must be the case that $|a_n| > 1$ (Why is this so?). Thus it will definitely happen that $\lim_{n\to\infty} a_n \neq 0$.

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- This test is great for series terms a_n involving powers of n. Then $\sqrt[n]{|a_n|}$ will strip it off. Indeed, if the inside of a term like $a_n = (\cdots)^n$ has a limit which is less than 1, then the series $\sum a_n$ will converge. As an example, show that the series
$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$$
converges. By the Root Test, it will. (Now try to show convergence using the3 Ration Test....)
- The Root Test is not so useful with terms involving factorials.
- As something to keep in mind, what is $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}}$? Here
$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln\left(n^{\frac{1}{n}}\right)} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim_{n\to\infty} \frac{\ln n}{n}} = e^0 = 1.$$
This will be very useful to remember for the following type of problem:

Example 2. *Test the following series for convergence:* $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(\ln n)^n}$.

Here, $a_n = \frac{n}{(\ln n)^n}$, and by the Root Test, we have
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{(\ln n)^n}\right|} = \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{\ln n} = \lim_{n\to\infty} n^{\frac{1}{n}} \cdot \lim_{n\to\infty} \frac{1}{\ln n},$$
where the last equal sign is only valid if the individual limits actually exist. They do since the first is 1 (see above), and the second is 0. Hence the limit of a product IS the product of the limits, and equals 0. Hence by the Root Test, the series converges.

Example 3. *How does the Root Test do with geometric series?*

Given $\sum a_n = \sum_{n=1}^{\infty} ar^{n-1}$, for some $a, r \in \mathbb{R}$, we first note that it will be helpful to us to change the subscript on the terms a bit. Notice that
$$\sum a_n = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n.$$
This does not change the series, but it does make it a bit easier to manage. Then, by the Root Test, we get
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|ar^n|} = \lim_{n\to\infty} \sqrt[n]{|a|} \cdot |r| = |r| \lim_{n\to\infty} \sqrt[n]{|a|} = |r| \cdot 1 = |r|.$$
(This last limit equals 1, as long as $a \neq 0$. Why is this so?) And by the Root Test, as long as this limit is less than 1, the series will converge. Hence we see that a geometric series will converge as long as $|r| < 1$, just like before.

We have a definition:

Definition 4. A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

where each c_i, $i \in \mathbb{N}$, is a constant, is called a *power series*. The c_i's are called *coefficients*.

Some notes:

- The constants are given in any specific series and often are functions of n.
- For any given value for x, we can ask whether the series converges or not.
- In fact, whenever the series converges, the series has a sum. For different values of x where the series converges, this sum may be different. This assignment of a value of x to the actual sum of the convergent series is a function, and

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

on the domain of values where the series converges. One can think of this series as an infinite degree polynomial defined on the domain where it makes sense.

Example 5. Let $c_i = 1$ for all $i = 0, 1, 2, 3, \ldots$. Then

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots .$$

This is a geometric series with $r = x$ and $a = 1$. Indeed, the standard geometric series is $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$. Rewrite it as

$$f(x) = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} x^n.$$

We know precisely when the geometric series will converge. namely, when $|r| < 1$, which in this case means $|x| < 1$. We also know, when the series converges, what its sum is: $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. In our case we get

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \qquad \text{for all } x \in (-1, 1).$$

- A more general form for a power series is
$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$
called a *power series centered at a* or a *power series at a*. Note that the original definition of a power series is of one centered at $x = 0$. Also note that centering a power series about a means that the term inside the parentheses is 0 precisely when $x = a$.
- All tests are valid for determining when a power series converges. They are used to find values for x where the series converges. This is important since we will use power series as a substitute for functions when they converge.

Example 6. For what values of x does the following series converge: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$?

Let's use the Ratio Test here, where $a_n = \frac{(x-5)^n}{n}$ and $a_{n+1} = \frac{(x-5)^{n+1}}{n+1}$. Then
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-5)^{n+1}}{n+1}}{\frac{(x-5)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{n+1}\cdot\frac{n}{(x-5)^n}\right|$$
$$= \lim_{n\to\infty}|x-5|\frac{n}{n+1} = |x-5|\lim_{n\to\infty}\frac{n}{n+1} = |x-5|\cdot 1 = |x-5|.$$
Since a series will converge if this limit using the Ratio Test is strictly less than 1, we will have that as long as $|x-5| < 1$, the series will converge. But that means that $-1 < x-5 < 1$, or $4 < x < 6$. Note that by the Root Test, we will get the same result. Try this now.

Example 3 on page 742 of the text is a very nice example. Follow it closely. We will do more next time.