EXAMPLE: SECTION 9.5: LINEAR DIFFERENTIAL EQUATIONS

110.109 CALCULUS II (PHYS SCI & ENG)
PROFESSOR RICHARD BROWN

Problem. Exercise 9.5.15. Solve the Initial Value Problem $x^2y' + 2xy = \ln x$, $y(1) = 2$.

Strategy. This IVP has a linear, first order ODE. We place the ODE in its standard form to compute the integrating factor. Then we multiply the ODE by its integrating factor and then integrate to find the general solution. We then use the initial data to find the particular solution.

Solution. This ODE is almost in its standard form. To get there, simply divide already in its standard form, we can read off the function $P(x)$ as the coefficient of the term involving the y variable. So divide the ODE by x^2 to get

$$\begin{aligned} \frac{1}{x^2}\left[x^2y' + 2xy \,=\, \ln x\right] \\ y' + \frac{2}{x}y = \frac{\ln x}{x^2}. \end{aligned}$$

Here then, $P(x) = 2x$, and the integrating factor is

$$e^{\int P(x)\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = e^{\ln x^2} = x^2.$$

Multiply this through the ODE to get

$$\begin{aligned} x^2\left[y' + \frac{2}{x}y \,=\, \frac{\ln x}{x^2}\right] \\ x^2y' + 2xy = \ln x. \end{aligned}$$

Note. *Okay, you are thinking that you are right back where you started from, and didn't need to do this part at all! You are right, in that the left-hand-side of the original ODE was already in the nice form as the total differential of a product of functions. However, if you didn't see that before, you do now. The step didn't lead you astray. Rather, it simply showed that, in this particular case, the step was superfluous.*

Continuing on, we get

$$\begin{aligned} x^2y' + 2xy &= \ln x. \\ \frac{d}{dx}\left[x^2y\right] &= \ln x. \\ \int \frac{d}{dx}\left[x^2y\right]\,dx &= \int \ln x\,dx \\ x^2y &= x\ln x - x + C \\ y(x) &= \frac{\ln x}{x} - \frac{1}{x} + \frac{C}{x^2}. \end{aligned}$$

This last equation is the general solution. To find the particular solution, note that

$$y(1) = \frac{\ln 1}{1} - \frac{1}{1} + \frac{C}{1^2} = -1 + C = 2,$$

so that $C = 3$. Hence the particular solution to the IVP is

$$y = \frac{\ln x}{x} - \frac{1}{x} + \frac{3}{x^2}.$$

Does this work? We check: Note the for the given solution $y(x)$, we have

$$y'(x) = -\frac{1}{x^2}\ln x + \frac{1}{x^2} - \left(-\frac{1}{x^2}\right) - \frac{6}{x^3} = -\frac{1}{x^2}\ln x + \frac{2}{x^2} - \frac{6}{x^3}.$$

With the substitution of both $y(x)$ and $y'(x)$ into the ODE, we get

$$\begin{aligned} x^2y' + 2xy &= \ln x \\ x^2\left(-\frac{1}{x^2}\ln x + \frac{2}{x^2} - \frac{6}{x^3}\right) + 2x\left(\frac{\ln x}{x} - \frac{1}{x} + \frac{3}{x^2}\right) &= \ln x \\ -\ln x + 2 - \frac{6}{x} + 2\ln x - 2 + \frac{6}{x} &= \ln x \\ \ln x &= \ln x. \end{aligned}$$

So it all works.