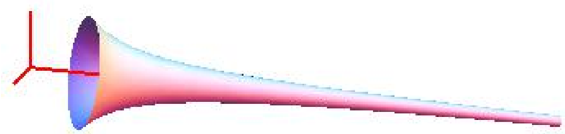
EXAMPLE: SECTION 7.8: IMPROPER INTEGRALS: GABRIEL'S HORN

110.109 CALCULUS II (PHYS SCI & ENG)
PROFESSOR RICHARD BROWN

Construct the surface of revolution given by rotating the function $f(x) = \frac{1}{x}$ on $[1, \infty)$ around the x-axis (see the figure). This surface is called Gabriel's Horn, or Gabriel's Trumpet, due to a highly unusual and paradoxical trait (we, as mathematicians, tend to name those structures that provide great insight into the more subtle points of our constructions). Let's study this shape in terms of its mathematical construction: We start with a general question: How much space lies within the Horn?



Question 1. *What is the volume enclosed by this shape on the interval* $[1, \infty)$.

The volume of a surface of revolution given by rotating the function $f(x)$, defined on the interval $[a, b]$, around the x-axis is

$$V = \int_a^b A(x)\, dx,$$

where $A(x)$ is the cross-sectional area at $x \in [a, b]$. Due to the construction, this is ALWAYS a circle, of radius $f(x)$, and hence $A(x) = \pi\,(f(x))^2$, so that

$$V = \int_a^b \pi\,(f(x))^2\, dx.$$

In our case, the interval of integration is infinite, and hence the integral we define is *improper*. Nevertheless, we find that

$$\begin{aligned} \text{vol(Gabriel's Horn)} &= \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^\infty \frac{dx}{x^2} \\ &= \pi \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx = \pi \lim_{b\to\infty} \left(-\frac{1}{x}\bigg|_1^b \right) \\ &= \pi \lim_{b\to\infty} \left(-\frac{1}{b} + 1 \right) = \pi(1) = \pi. \end{aligned}$$

Hence, even though the Horn extends outward along the x-axis to ∞, the improper integral does converge, and hence there is finite volume "inside" the Horn. One can say that one can fill the Horn with π-units of a liquid....

Another facet of the Horn to study is how much material must it take to construct the Horn.

Question 2. *What is the surface area of Gabriel's Horn?*

The surface area of a surface of revolution is the subject of Section 8.2. For a surface formed by revolving $f(x)$ on $[a, b]$ around the x-axis, the surface area is found by evaluating

$$SA = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

This formula basically says that one can find surface area by multiplying the circumference of the surface of evolution at x, which is a circle again, with circumference $2\pi f(x)$, by the arc-length along the original function $f(x)$ (this is the radical part of the integrand). In our case, we get an improper integral again (call the surface area SA):

$$SA(\text{Gabriel's Horn}) = \int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \left[-\frac{1}{x^2}\right]^2}\, dx = 2\pi\int_1^\infty \left(\frac{1}{x}\right)\sqrt{1 + \frac{1}{x^4}}\, dx.$$

This integral is not such an easy calculation. However, we really do not need to actually calculate this quantity using an antiderivative. Instead, we make the following observation:

Notice that on the interval $[1, \infty)$, we have that

$$\sqrt{1 + \frac{1}{x^4}} > \sqrt{1} = 1.$$

Thus we can say that, if

$$g(x) = 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \frac{1}{x^4}} > 2\pi\left(\frac{1}{x}\right) = h(x) > 0$$

on the interval $[1, \infty)$, then

$$\int_1^\infty g(x)\, dx > \int_1^\infty h(x)\, dx$$

by the properties of integrals. And by the Comparison Theorem for improper integrals, we can conclude that, if the integral of the smaller one (with $h(x)$ as the integrand) diverges, then so does the integral of the larger function $g(x)$. Indeed, we find by comparison that

$$\int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \frac{1}{x^4}}\, dx > \int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1}\, dx = 2\pi\int_1^\infty \frac{1}{x}\, dx.$$

But we have already evaluated this last integral in class. We get

$$2\pi\int_1^\infty \frac{1}{x}\, dx = 2\pi\lim_{b\to\infty}\int_1^b \frac{1}{x}\, dx = 2\pi\lim_{b\to\infty}\left(\ln|x|\Big|_1^b\right) = 2\pi\lim_{b\to\infty}\ln b = \infty.$$

Hence this last integral diverges, and hence by comparison so does the former integral. But this implies that the surface area of Gabriel's Horn is infinite!

$$SA(\text{Gabriel's Horn}) = \int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \left[-\frac{1}{x^2}\right]^2}\, dx = \infty.$$

So we have a surface with infinite surface area enclosing a finite volume. In essence, we have a "bucket" that would take an infinite amount of material to make, but which holds a finite amount of stuff. A paradox? Think of it this way: We could fill Gabriel's Horn with π-units of paint. We could then empty the Horn, leaving the paint residue along the sides. But we could never actually pant Gabriel's Horn using a finite amount of paint!

So what gives here????