EXAMPLE: SECTION 7.4: PARTIAL FRACTION DECOMPOSITION

110.109 CALCULUS II (PHYS SCI & ENG)
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Question 1. *Calculate* $\displaystyle\int \frac{3x^2-1}{(x^2+4)(x-3)}\,dx$.

Note. *Here is an example of an integral of a rational function whose denominator (already factored completely) contains a quadratic term factor. The partial fraction decomposition is a little different for quadratic terms.*

Proof. We do a partial fraction decomposition on the integrand. Here write

$$\frac{3x^2-1}{(x^2+4)(x-3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-3}.$$

We need two undetermined constants in the first fraction due to the fact that all of the summands in a partial fraction decomposition are proper. But since we cannot factor the quadratic in the first denominator, the numerator can either be a constant term or a linear term. To consider ANY linear term, note that ALL possible linear and constant terms are of the form $Ax + B$, where A and B are unknown. This will include all possible constant term numerators (if we eventually find out that $A = 0$). Now recombine the fractions on the right so that the resulting fraction has the same denominator as the left-hand side:

$$\begin{aligned}
\frac{3x^2-1}{(x^2+4)(x-3)} &= \frac{Ax+B}{x^2+4} + \frac{C}{x-3} \\
&= \frac{(Ax+B)(x-3)}{(x^2+4)(x-3)} + \frac{C(x^2+4)}{(x-3)(x^2+4)} \\
&= \frac{Ax^2-3Ax+Bx-3B+Cx^2+4C}{(x^2+4)(x-3)} \\
&= \frac{(A+C)x^2+(B-3A)x+(4C-3B)}{(x^2+4)(x-3)}.
\end{aligned}$$

Since the denominators are the same, the numerators must be the same also. And whenever two polynomials are equal, it must be the case that all corresponding coefficients must be equal. This give us a set of equations with which we can solve for the unknowns, Here we get

$$\begin{aligned}
3 &= A+C \\
0 &= B-3A \\
-1 &= 4C-3B.
\end{aligned}$$

We can solve this immediately as follows: The middle equations says that $B = 3A$. With a substitution into the first equation, we get the 2 equations in 2 unknowns:

$$\begin{aligned}
3 &= C+A \\
-1 &= 4C-9A,
\end{aligned}$$

which, upon multiplying the first equation by 9 and adding the two equations together, results in the single equation $26 = 13C$, or $C = 2$. With $C = 2$, we get $A = 1$ from the first equation, and

then $B = 3$ from the middle one. Thus
$$\frac{3x^2-1}{(x^2+4)\,(x-3)} = \frac{x+3}{x^2+4} + \frac{2}{x-3},$$
and
$$\int \frac{3x^2-1}{(x^2+4)\,(x-3)}\,dx = \int \frac{x+3}{x^2+4}\,dx + \int \frac{2}{x-3}\,dx = \int \frac{x}{x^2+4}\,dx + \int \frac{3}{x^2+4}\,dx + \int \frac{2}{x-3}\,dx.$$

The last integral is quite doable, and immediately, we get (we ignore the constant at the moment)
$$\int \frac{2}{x-3}\,dx = 2\ln|x-3|.$$
The first is computable with the substitution $u = x^2+4$, $du = 2x\,dx$, or $\frac{1}{2}du = x\,dx$. Then
$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln\left|x^2+4\right|.$$
For the last one, we go back to Section 7.3, and with the form below to write $3\int \frac{1}{x^2+4}\,dx = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right)$. Put it all together and we get
$$\int \frac{3x^2-1}{(x^2+4)\,(x-3)}\,dx = \frac{1}{2}\ln\left|x^2+4\right| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + 2\ln|x-3| + C.$$

Let's check to see if this is correct: Write
$$\begin{aligned}
\frac{d}{dx}\left[\frac{1}{2}\ln\left|x^2+4\right| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + 2\ln|x-3| + C\right] &= \frac{1}{2}\left(\frac{1}{x^2+4}\right)2x + \frac{3}{2}\left(\frac{1}{1+\left(\frac{x}{2}\right)^2}\right) + \frac{2}{x-3}\\
&= \frac{x}{x^2+4} + \frac{3}{4+x^2} + \frac{2}{x-3}\\
&= \frac{x(x-3)}{(x^2+4)(x-3)} + \frac{3(x-3)}{(x^2+4)(x-3)} + \frac{2(x^2+4)}{(x^2+4)(x-3)}\\
&= \frac{x^2-3x+3x-9+2x^2+8}{(x^2+4)(x-3)}\\
&= \frac{3x^2-1}{(x^2+4)(x-3)}.
\end{aligned}$$
$\square$