EXAMPLE: SECTION 7.3: TRIGONOMETRIC SUBSTITUTION

110.109 CALCULUS II (PHYS SCI & ENG)
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Question 1. *Calculate* $\displaystyle\int_0^{\frac{3}{2}} \frac{1}{(4x^2+9)^{\frac{3}{2}}}\,dx$.

Strategy. *Here is an example of an integral whose integrand has the structure of one of the three forms found in Section 7.3. In this case, one would naturally think that the substitution $x = a\tan\theta$ is a good one due to the sum of the terms under the radical. However, the choice of the constant a needs to be studied. The whole point of the substitution, really, is to change the sum under the radical (inside the parentheses) into a perfect square to kill off the fractional power. Finding a way to do that is the reason I am offering this problem.*

Solution. Rewrite the integrand

$$\frac{1}{(4x^2+9)^{\frac{3}{2}}} = \frac{1}{\left(\sqrt{4x^2+9}\right)^3}$$

as a way to "see" the form better. Since the expression under the radical is a sum, the form is much like the one that benefits from the substitution $x = a\tan\theta$. But the point of the substitution is to find a way to make what is under the radical

$$4x^2+9 = 4a^2\tan^2\theta + 9 = 9\tan^2\theta + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta.$$

Hence you should choose your value for a so that $4a^2 = 9$. Thus, $a = \frac{3}{2}$ is a good choice.

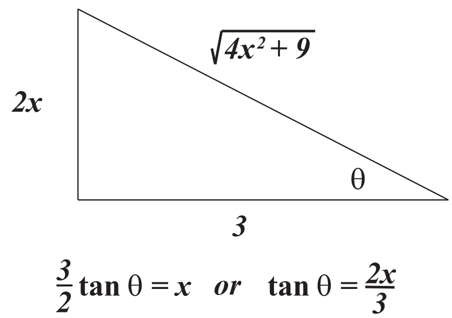
Let $x = \frac{3}{2}\tan\theta$. Then $dx = \frac{3}{2}\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{(4x^2+9)^{\frac{3}{2}}}\,dx = \int \frac{1}{\left(\sqrt{4x^2+9}\right)^3}\,dx &= \int \frac{1}{\left(\sqrt{4\left(\frac{4}{9}\tan^2\theta\right)+9}\right)^3}\frac{3}{2}\sec^2\theta\,d\theta \\ &= \int \frac{1}{(3|\sec\theta|)^3}\frac{3}{2}\sec^2\theta\,d\theta\end{aligned}$$

Now, since this is a definite integral, we can use the substitution to change the limits. Then we will not have to "go back" to the variable x. When $x = 0$, this means that $\frac{3}{2}\tan\theta = 0$ which is solved by $\theta = 0$. When $x = \frac{3}{2}$, we get $\frac{3}{2} = \frac{3}{2}\tan\theta$, or $\tan\theta = 1$, which is solved by $\theta = \frac{\pi}{4}$. And on this interval, $\sec\theta > 0$. Hence the absolute value signs are not needed, and

$$\begin{aligned}\int_0^{\frac{3}{2}} \frac{1}{(4x^2+9)^{\frac{3}{2}}}\,dx &= \int_0^{\frac{\pi}{4}} \frac{1}{(3\sec\theta)^3}\frac{3}{2}\sec^2\theta\,d\theta \\ &= \frac{3}{54}\int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta}\,d\theta = \frac{1}{18}\int_0^{\frac{\pi}{4}} \cos\theta\,d\theta = \frac{1}{18}\sin\theta\Big|_0^{\frac{\pi}{4}} = \frac{1}{18}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{36}.\end{aligned}$$

Now, we cannot really go backwards here directly to check since we never actually found the antiderivative of the original integrand as a function of x. We did get that the integrand changed from $\frac{1}{(4x^2+9)^{\frac{3}{2}}}$ to $\frac{1}{18}\cos\theta$, whose antiderivative is $\frac{1}{18}\sin\theta$. Switching this back to the x variable is effectively the antiderivative of the original integrand. We use the triangle below to find this. Given this triangle, it is easy to see that $\sin\theta = \frac{2x}{\sqrt{4x^2+9}}$, so that $\frac{1}{18}\sin\theta = \frac{x}{9\sqrt{4x^2+9}}$. Hence we see that



$$\int \frac{1}{(4x^2+9)^{\frac{3}{2}}}\,dx = \frac{x}{9\sqrt{4x^2+9}} + C.$$

So to check that our calculations were correct, we check that this last assertion is true.

Here

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x}{9\sqrt{4x^2+9}} + C\right] &= \frac{1}{9}\cdot\frac{d}{dx}\left[\frac{x}{\sqrt{4x^2+9}} + C\right] \\
&= \frac{1}{9}\cdot\frac{\left(\sqrt{4x^2+9} - x\left(\frac{1}{2\sqrt{4x^2+9}}\right)8x\right)}{4x^2+9} \\
&= \frac{1}{9}\cdot\frac{\left(\frac{4x^2+9}{\sqrt{4x^2+9}} - \frac{4x^2}{\sqrt{4x^2+9}}\right)}{\left(\sqrt{4x^2+9}\right)^2} \\
&= \frac{1}{9}\cdot\frac{9}{(4x^2+9)^{\frac{3}{2}}} \\
&= \frac{1}{(4x^2+9)^{\frac{3}{2}}}.
\end{aligned}$$

Question 2. *Find an antiderivative of* $f(x) = \dfrac{1}{\sqrt{x^2+2x}}$.

Strategy. *Here is an example of an integral whose integrand again kind of has the structure of one of the three forms found in Section 7.3, but must be manipulated first. In this case, one can "complete the square" to make what is under the radical better suited to the forms.*

Solution. Any quadratic polynomial can be written as a sum or difference of squares through the process of completing the square. Here

$$x^2+2x = x^2+2x+1-1 = (x^2+2x+1)-1 = (x+1)^2-1.$$

Thus

$$\int \frac{1}{\sqrt{x^2+2x}}\,dx = \int \frac{1}{\sqrt{(x+1)^2-1}}\,dx = \int \frac{1}{\sqrt{u^2-1}}\,du$$

after the substitution $u = x_1$, $du = dx$. This form benefits from the trigonometric substitution $u = \sec\theta$, where $du = \sec\theta\tan\theta\,d\theta$, and

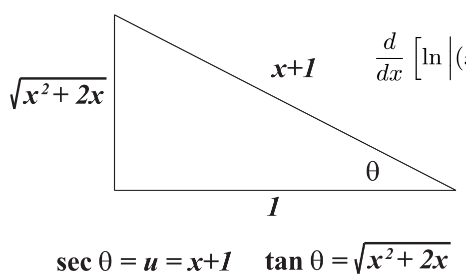
$$\int \frac{1}{\sqrt{u^2-1}}\,du = \int \frac{1}{\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta = \int \frac{1}{|\tan\theta|}\sec\theta\tan\theta\,d\theta.$$

Now here, we do not know before hand which interval we want to define our trig functions. The good part about that is that we are only looking for ANY antiderivative of $f(x)$. Hence we have the freedom to choose our interval and can can choose an interval where $\tan\theta$ is positive. Specifically, we can choose the interval $\left(0, \frac{\pi}{2}\right)$ where both $\sec\theta$ and $\tan\theta$ are positive. But really, we do not have to specify at all. Simply make a choice for $\tan\theta$. Then

$$\int \frac{1}{|\tan\theta|}\sec\theta\tan\theta\,d\theta = \int \frac{1}{\tan\theta}\sec\theta\tan\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C.$$

Again, using the substitution to help make the triangle below, we get

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2x}}\,dx = \int \frac{1}{\sqrt{u^2-1}}\,du = \int \sec\theta\,d\theta &= \ln|\sec\theta+\tan\theta| + C \\ &= \ln\left|u+\sqrt{u^2-1}\right| + C \\ &= \ln\left|(x+1)+\sqrt{x^2+2x}\right| + C.\end{aligned}$$



Does this work? Well, here

$$\begin{aligned}\frac{d}{dx}\left[\ln\left|(x+1)+\sqrt{x^2+2x}\right| + C\right] &= \frac{1+\frac{2x+2}{2\sqrt{x^2+2x}}}{(x+1)+\sqrt{x^2+2x}} \\ &= \frac{\frac{\sqrt{x^2+2x}+(x+1)}{\sqrt{x^2+2x}}}{(x+1)+\sqrt{x^2+2x}} \\ &= \frac{1}{\sqrt{x^2+2x}}.\end{aligned}$$