EXAMPLE: SECTION 7.2: TRIGONOMETRIC INTEGRALS

110.109 CALCULUS II (PHYS SCI & ENG)
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Question 1. Calculate $\int \sin^2 x \cos^4 x \, dx$.

Strategy. *This is an example of a trigonometric Integral of the form* $\displaystyle\int \sin^n x \cos^m x \, dx$. *When one of* $n, m \in \mathbb{N}$ *is odd, stripping off one of the odd factors to set up a simple substitution is detailed in the book. However, when both* m *and* n *are even, one must be a bit more clever. Here, we make use of the reduction formula for the antiderivative of positive powers of* $\sin x$.

Solution. Using the identity $\sin^2 x + \cos^2 x = 1$, we can rewrite this integral is a sum of integrals of powers of the sine function:

$$\begin{aligned}\int \sin^2 x \cos^4 x \, dx &= \int \sin^2 x \left(1 - \sin^2 x\right)^2 \, dx \\ &= \int \sin^2 x \left(1 - 2\sin^2 x + \sin^4 x\right) \, dx \\ &= \int \sin^2 x \, dx - 2\int \sin^4 x \, dx + \int \sin^6 x \, dx.\end{aligned}$$

This is helpful since the reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \int \sin^{n-2} x \, dx$$

allows us to "reduce" the degree of the integrand using the Integration by Parts technique. Doing this a few times will allow us to actually come up with a nice expression for the anti-derivative of the function $\sin^2 x \cos^4 x$. The first (and best) step would be to use the reduction formula on the largest degree integrand (why?). This way, the "reduced" expression will have an integrand similar to another summand in your expression. You can then combine them. So

$$\begin{aligned}\int \sin^2 x \cos^4 x \, dx &= \int \sin^2 x \, dx - 2\int \sin^4 x \, dx + \int \sin^6 x \, dx \\ &= \int \sin^2 x \, dx - 2\int \sin^4 x \, dx + \left(-\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} \int \sin^4 x \, dx\right) \\ &= \int \sin^2 x \, dx - \frac{7}{6} \int \sin^4 x \, dx - \frac{1}{6} \sin^5 x \cos x.\end{aligned}$$

Do you see why this works, and how your life is easier now? The integrand to the 6th power is gone. We now continue by reducing the next right-most integrand: So

$$\begin{aligned}
\int \sin^2 x \cos^4 x\, dx &= \int \sin^2 x\, dx - \frac{7}{6}\int \sin^4 x\, dx - \frac{1}{6}\sin^5 x \cos x \\
&= \int \sin^2 x\, dx - \frac{7}{6}\left(-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\int \sin^2 x\, dx\right) - \frac{1}{6}\sin^5 x \cos x \\
&= \int \sin^2 x\, dx + \frac{7}{24}\sin^3 x \cos x - \frac{21}{24}\int \sin^2 x\, dx - \frac{1}{6}\sin^5 x \cos x. \\
&= \frac{3}{24}\int \sin^2 x\, dx + \frac{7}{24}\sin^3 x \cos x - \frac{1}{6}\sin^5 x \cos x \\
&= \frac{3}{24}\left(-\frac{1}{2}\sin x \cos x + \frac{1}{2}\int \sin^0 x\, dx\right) + \frac{7}{24}\sin^3 x \cos x - \frac{1}{6}\sin^5 x \cos x \\
&= \frac{3}{48}x - \frac{3}{48}\sin x \cos x + \frac{7}{24}\sin^3 x \cos x - \frac{1}{6}\sin^5 x \cos x + C.
\end{aligned}$$

Now, as a check, we show that

$$\frac{d}{dx}\left[\frac{3}{48}x - \frac{3}{48}\sin x \cos x + \frac{7}{24}\sin^3 x \cos x - \frac{1}{6}\sin^5 x \cos x + C\right] = \sin^2 x \cos^4 x.$$

Here

$$\begin{aligned}
\frac{d}{dx}&\left[\frac{3}{48}x - \frac{3}{48}\sin x \cos x + \frac{7}{24}\sin^3 x \cos x - \frac{1}{6}\sin^5 x \cos x + C\right] \\
&= \frac{3}{48} - \frac{3}{48}\left(\cos^2 x - \sin^2 x\right) + \frac{7}{24}\left(3\sin^2 x \cos^2 x - \sin^4 x\right) - \frac{1}{6}\left(5\sin^4 x \cos^2 x - \sin^6 x\right) \\
&= \frac{3}{48} - \frac{3}{48}\left(1 - 2\sin^2 x\right) + \frac{7}{24}\left(3\sin^2 x - 4\sin^4 x\right) - \frac{1}{6}\left(5\sin^4 x - 6\sin^6 x\right) \\
&= \frac{3}{48} - \frac{3}{48} + \frac{6}{48}\sin^2 x + \frac{21}{24}\sin^2 x - \frac{28}{24}\sin^4 x - \frac{5}{6}\sin^4 x + \sin^6 x \\
&= \sin^2 x - 2\sin^4 x + \sin^6 x \\
&= \sin^2 x\left(1 - 2\sin^2 x + \sin^4 x\right) \\
&= \sin^2 x\left(1 - \sin^2 x\right)^2 \\
&= \sin^2 x \cos^4 x.
\end{aligned}$$