EXAMPLE: SECTION 7.1: INTEGRATION BY PARTS

110.109 CALCULUS II (PHYS SCI & ENG)
PROFESSOR RICHARD BROWN

Question 1. *Prove the reduction formula* $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$ *for* $n > 1$.

Strategy: Here, we will use the Integration by Parts method (IbP) to rewrite the integrand as a product of functions be stripping off one of the factors in the power. Then the right-hand-side integral in the IbP will still only involve trig functions. We can see this due to the pattern in the first function after the equal sign in the reduction formula above. (when trying to prove a formula, looking for patterns and structure in the formula itself is not cheating. It is the proper exploration of structure.) After utilizing the IbP method, we will use algebra to hopefully expose the formula.

Solution: First, notice that if we try the assignments $u = \sin^n x$ and $dv = 1dx$, then $du = n\sin^{n-1} x\cos x$ and $v = x$. Then the IbP gives us the following:

$$\int \sin^n x\,dx = x\sin^n x - \int nx\sin^{n-1} x\cos x\,dx.$$

While this is certainly correct, the introduction of an x into the integrand on the right-hand side, along with the now two trig functions has made the problem more complicated to solve. Instead of continuing this path, let's try a different approach.

Instead, since $n > 1$, strip off one of the factors and write the integrand as a product of two trig functions, one to the power $n-1$. Then at least after we apply the IbP, the right-hand-side integral will have only trig functions in it. Hence we let $u = sin^{n-1}x$ and $dv = \sin x\,dx$, so that $du = (n-1)\sin^{n-2} x\cos x\,dx$, and $v = -\cos x$. The IbP formula then becomes

$$\begin{aligned}\int \sin^n x\,dx = \int \sin^{n-1} x\sin x\,dx &= -\sin^{n-1} x\cos x - \int (-\cos x)(n-1)\sin^{n-2} x\cos x\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx.\end{aligned}$$

Notice that in this last integral, there is no x as a polynomial present. In fact, using the identity $\sin^2 x + \cos^2 x = 1$, or $\cos^2 x = 1-]sin^2 x$, we get

$$\begin{aligned}\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.\end{aligned}$$

We now have two terms that include the integral of $\sin^n x$, then thing we are hoping solve for. Using just a bit of algebra, we can combine them:

$$\begin{aligned}
\int \sin^n x\,dx + (n-1)\int \sin^n x\,dx &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx \\
(1+n-1)\int \sin^n x\,dx &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx \\
n\int \sin^n x\,dx &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx \\
\int \sin^n x\,dx &= -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.
\end{aligned}$$