EXAMPLE: SECTION 7.1: INTEGRATION BY PARTS (THE ANTI-PRODUCT RULE)

110.109 CALCULUS II (PHYS SCI & ENG) PROFESSOR RICHARD BROWN

Question 1. Calculate $\int \sin^{-1} x \, dx$.

Note. This is an example of a integral does not have the form of a product of functions. However, by thinking of one of the functions as a constant 1, the Integration by Parts procedure allows us to instead integrate the derivative of the inverse sine function. This will be easier. One other point is that this function $\sin^{-1} x$ is NOT "1 over the sine function", or

$$\sin^{-1} x \neq \frac{1}{\sin x} = \sec x.$$

This is a common mistake due to ambiguous notation. Rather, it is the function-inverse of the sine function, as in $y = \sin^{-1} x$ is the inverse function of $\sin y = x$. To solve this, we will first compute the derivative of $\sin^{-1} x$. Then we will use this fact.

Proof. To compute the derivative of the function $y = \cos^{-1} x$, we will instead use the inverse of this function $\sin y = x$ and use implicit differentiation. Here then thinking of y as an implicit function of x, we have

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$
$$(\cos y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

This is not quite complete, as we have only expressed the derivative of $y = \sin^{-1} x$ in terms of the function y. We can fix this through an understanding of how triangles relate the trigonometric functions of their angles to their side lengths and such. Form a right triangle with one angle y and hypotenuse 1. Then the side adjacent to the angle y has length $\cos y$ and the side opposite to the angle y has length $\sin y$. Use the picture at the end as a guide, and notice that the opposite side $\sin y = x$. The Pythagorean's Theorem gives us that $x^2 + \cos^2 y = 1$, or $\cos y = \sqrt{1 - x^2}$. Thus

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sin^{-1} x \right) = \frac{1}{\sqrt{1 - x^2}}.$$

Why was this helpful? Now use Integration by Parts to rewrite the integral. Here, let $f(x) = \sin^{-1} x$, and g(x) = 1 in the formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Then $f'(x) = \frac{1}{\sqrt{1-x^2}}$, and g(x) = x, and we get

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, dx$$

To finish the calculation, notice that the integral on the right hand side of the last equation can be solved with a straightforward substitution: Let $u = 1 - x^2$, so that $du = -2x \, dx$, or $-\frac{1}{2} \, du = x \, dx$. Then

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, dx$$
$$= x \sin^{-1} x - \int \frac{-1}{2\sqrt{u}} \, du$$
$$= x \sin^{-1} x + \sqrt{u} + C = x \sin^{-1} x + \sqrt{1 - x^2} + C.$$

Is this correct? Let's check by showing that $\frac{d}{dx} \left[x \sin^{-1} x + \sqrt{1 - x^2} + C \right] = \sin^{-1} x$. Here

