

Name: _____

110.211 HONORS MULTIVARIABLE CALCULUS
SPRING 2008
MIDTERM EXAMINATION
April 28, 2008

Instructions: The exam is **7** pages long, including this title page. The last page is a page of formulae you may find useful on the exam. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		15
2		20
3		25
4		20
5		20
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: _____ Date: _____

Question 1. [15 points] Verify or disprove the following:

(a) There exists a C^2 function $f(x, y, z)$ whose gradient is the vector field

$$\vec{F} = (e^x \cos y + e^{-x} \sin z) \vec{i} - e^x \sin y \vec{j} - e^{-x} \cos z \vec{k}.$$

(b) There exists a C^2 vector field $\vec{G}(x, y, z)$ whose curl is the vector field

$$\vec{F} = x(y^2 + 1) \vec{i} + (ye^x - e^z) \vec{j} + x^2 e^z \vec{k}.$$

Solutions:

(a) Let $f(x, y, z) = e^x \cos y - e^{-x} \sin z$. Then $\vec{F} = \nabla f$.

(b) If $\vec{F} = \nabla \times \vec{G}$ for some C^2 vector field $\vec{G}$, it would have to be the case that $\nabla \cdot \vec{F} = 0$, since the curl of a vector field is incompressible as a vector field. But

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial}{\partial x} (x(y^2 + 1)) + \frac{\partial}{\partial y} (ye^x - e^z) + \frac{\partial}{\partial z} (x^2 e^z) \\ &= y^2 + 1 + e^x + x^2 e^z \neq 0. \end{aligned}$$

Hence, it is NOT the case that $\vec{F}$ is the curl of some other vector field $\vec{G}$.

Question 2. [20 points] Use the method of Lagrange multipliers to find the minimum distance from the origin to the surface $x^2 - (y - z)^2 = 1$.

Solution:

The method of Lagrange multipliers leads to the system given by the vector equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, and the scalar equation $g(x, y, z) = 1$, where $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ and $g(x, y, z) = x^2 - (y - z)^2$. The system is:

$$\begin{aligned}\frac{x}{\sqrt{x^2 + y^2 + z^2}} &= \lambda 2x \\ \frac{y}{\sqrt{x^2 + y^2 + z^2}} &= -\lambda 2(y - z) \\ \frac{z}{\sqrt{x^2 + y^2 + z^2}} &= \lambda 2(y - z) \\ x^2 - (y - z)^2 &= 1.\end{aligned}$$

Notice that the first equation leads immediately to the multiplier: $\lambda = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}$. With this, the system boils down to

$$\begin{aligned}x &= x \\ y &= -(y - z) \\ z &= y - z \\ x^2 - (y - z)^2 &= 1.\end{aligned}$$

(we could have immediately “chosen” our distance function without the square root sign; This would have made λ “nicer” and the boiled down system would have been the same. You should think about why this works?). The middle two equations are satisfied only for $y = z = 0$ (their graphs are two lines in the yz -plane that cross at the origin), and these values coupled with the last equation and the first yield $x = \pm 1$.

To see that these critical points $(-1, 0, 0)$ and $(1, 0, 0)$ must be minima (recall that for functions of more than one variable, one can have more than one local minima without having a local maxima: See Problem 47, p. 259 in the text), look directly at the distance function. For a point along the surface, the x -coordinate must satisfy $x^2 = 1 + (y - z)^2$. Hence

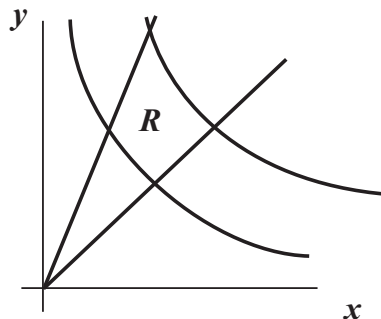
$$f(x, y, z) \Big|_{g(x,y,z)=1} = \sqrt{x^2 + y^2 + z^2} \Big|_{x^2=1+(y-z)^2} = \sqrt{1 + (y - z)^2 + y^2 + z^2} \geq 1$$

everywhere on the surface. Hence any point of distance 1 from the origin is a minimum.

Question 3. [25 points] Let R be the region in $\mathbb{R}^2$ bounded by the lines $y = x$ and $y = 4x$, and the hyperbolas $xy = 1$ and $xy = 4$. And let R be the image of a domain D under the transformation $T(u, v) = (\frac{u}{v}, v)$.

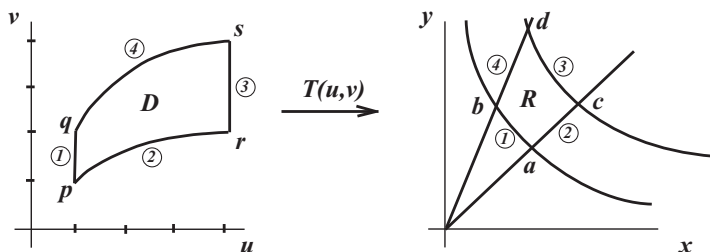
(a) Find and sketch D .

(b) Evaluate $\iint_R xy^3 dA$.



Solution:

(a) .



$$\begin{aligned} p = (1, 1) &\rightarrow (1, 1) = a \\ q = (1, 2) &\rightarrow (.5, 2) = b \\ r = (4, 2) &\rightarrow (2, 2) = c \\ s = (4, 4) &\rightarrow (1, 4) = d \end{aligned}$$

$$\begin{aligned} v &= y \\ u &= xy \end{aligned} \quad \rightarrow \quad \begin{aligned} \textcircled{2} \quad u &= v^2 \\ \textcircled{4} \quad 4u &= v^2 \end{aligned}$$

(b) Here we use the Change of Variables Formula directly:

$$\begin{aligned} \iint_R xy^3 dx dy &= \iint_D x(u, v) y^3(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \int_1^4 \int_{\sqrt{u}}^{2\sqrt{u}} \frac{u}{v} v^3 \left| \frac{1}{v} \right| dv du \\ &= \int_1^4 \int_{\sqrt{u}}^{2\sqrt{u}} uv dv du \\ &= \int_1^4 \left(u \frac{v^2}{2} \Big|_{\sqrt{u}}^{2\sqrt{u}} \right) du \\ &= \int_1^4 \frac{3}{2} u^2 du = \frac{u^3}{2} \Big|_1^4 = 32 - \frac{1}{2}. \end{aligned}$$

Question 4. [20 points] Do exactly ONE of the following:

- (a) Let C be a simple closed curve in $\mathbb{R}^2$. Show that the area of the region enclosed by C equals the circulation of the vector field $\vec{F} = \left(-\frac{y}{2}, \frac{x}{2}\right)$ along C .
- (b) Use Green's Theorem to prove the following special case of the Change of Variables Formula

$$\iint_D dx dy = \iint_{D^*} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

for the transformation $(u, v) \mapsto (x(u, v), y(u, v))$ (Hint: Expand the Jacobian. You will use Green's Theorem on each side).

Solutions:

- (a) The circulation of $\vec{F}$ is the line integral

$$\oint_C \vec{F} \cdot d\vec{s} = \oint_C -\frac{y}{2} dx + \frac{x}{2} dy.$$

By Green's Theorem for a C^1 vector field $\vec{F} = M(x, y)\vec{i} + N(x, y)\vec{j}$ defined on a region D in the plane bounded by a closed, simple curve $C = \partial D$,

$$\oint_C M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

Hence,

$$\begin{aligned} \oint_C -\frac{y}{2} dx + \frac{x}{2} dy &= \iint_D \left(\frac{\partial}{\partial x} \left(\frac{x}{2} \right) - \frac{\partial}{\partial y} \left(-\frac{y}{2} \right) \right) dA \\ &= \iint_D \left(\frac{1}{2} - \left(-\frac{1}{2} \right) \right) dA = \iint_D dA = \text{area}(D). \end{aligned}$$

- (b) Left as an exercise.

Question 5. [20 points] Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be C^2 on S , a compact, piecewise smooth orientable surface. Show $\oint_{\partial S} (f \nabla f) \cdot d\vec{s} = 0$.

Solution:

For f given as above, and $\vec{x} = (x, y, z)$, the integrand is a vector field:

$$f \nabla f(\vec{x}) = \left(f(\vec{x}) \frac{\partial f}{\partial x}(\vec{x}), f(\vec{x}) \frac{\partial f}{\partial y}(\vec{x}), f(\vec{x}) \frac{\partial f}{\partial z}(\vec{x}) \right).$$

Hence, this integral is a vector line integral about the closed curve ∂S . The surface S and the vector field satisfy Stoke's Theorem, so

$$\oint_{\partial S} (f \nabla f) \cdot d\vec{s} = \iint_S \nabla \times (f \nabla f) \cdot d\vec{S}.$$

For the first component of the vector field $\nabla \times (f \nabla f)$, we have

$$\begin{aligned} \nabla \times (f \nabla f) \cdot \vec{i} &= \frac{\partial}{\partial y} \left(f(\vec{x}) \frac{\partial f}{\partial z} \right) - \frac{\partial}{\partial z} \left(f(\vec{x}) \frac{\partial f}{\partial y} \right) \\ &= \left(\frac{\partial f}{\partial y} \frac{\partial f}{\partial z} + f(\vec{x}) \frac{\partial^2 f}{\partial y \partial z} \right) - \left(\frac{\partial f}{\partial z} \frac{\partial f}{\partial y} + f(\vec{x}) \frac{\partial^2 f}{\partial z \partial y} \right) = 0 \end{aligned}$$

since f is C^2 and mixed partials are equal. The other two component calculations similarly vanish.

Hence the vector field $\nabla \times (f \nabla f) = \vec{0}$ (the original vector field $f \nabla f$ is irrotational), and by Stoke's Theorem, the result follows.

Possibly helpful formulae

- $\iint_{\partial D} \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F} dV$
- $\iint_D \nabla \times \vec{F} d\vec{S} = \oint_{\partial D} \vec{F} \cdot d\vec{s}$
- $\iint_D \nabla \times \vec{F} \cdot \vec{k} dS = \oint_{\partial D} \vec{F} \cdot d\vec{s}$