Name: ______________________________ **Section Number:** ____________

110.211 HONORS MULTIVARIABLE CALCULUS
SPRING 2008
MIDTERM EXAMINATION
March 5, 2008

Instructions: The exam is **6** pages long, including this title page. The last page is a page of formulae you may find useful on the exam. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		25
2		15
3		20
4		25
5		15
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:** ____________

Question 1. [25 points] Let $F; \mathbb{R}^2 \to \mathbb{R}^2$ be given by the expression $F(x,y) = (\sin(xy), xy+y)$. Do the following:

(a) Compute the derivative of F.

(b) Determine whether it is possible to invert the system of equations $(u,v) = F(x,y)$ to solve for x and y as a function of u and v in a neighborhood of the point $(\pi, 1)$.

(c) Along the differential path in $\mathbb{R}^2$ given by $\mathbf{c}(t) = (t^2, 3t-2)$, write an expression for $\frac{dF}{dt}$ and evaluate $\left.\frac{dF}{dt}\right|_{t=\frac{2}{3}}$.

Solutions:

(a) $DF = \begin{bmatrix} y\cos(xy) & x\cos(xy) \\ y & x+1 \end{bmatrix}$.

(b) First, since the two component functions of F are C^1 (really, they are C^∞) on all of $\mathbb{R}^2$, so is F. Also,

$$DF\,(\pi,1) = \begin{bmatrix} \cos(\pi) & \pi\cos(\pi) \\ 1 & \pi+1 \end{bmatrix} = \begin{bmatrix} -1 & -\pi \\ 1 & \pi+1 \end{bmatrix}.$$

The determinant of this (which is the Jacobian of F at $(\pi,1)$) is

$$\det\, DF\,(\pi,1) = -\pi - 1 + \pi = 1 \neq 0.$$

Hence by the Inverse Function Theorem, there is a neighborhood of $(\pi,1)$ on which F is one-to-one onto its image and with a differentiable inverse. This means that there is a C^1 function G defined near $F(\pi,1) = (0, \pi+1)$ where $G(u,v) = (x,y)$.

(c) You can do this two ways (although there are the same way, really). One way is the following:

$$\begin{aligned}
\frac{dF}{dt} = D(F\circ\mathbf{c})(t) &= DF\,(\mathbf{c}(t))\,D\mathbf{c}(t) \\
&= \begin{bmatrix} y(t)\cos\left(x(t)y(t)\right) & x(t)\cos\left(x(t)y(t)\right) \\ y(t) & x(t)+1 \end{bmatrix} \begin{bmatrix} 2t \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} (3t-2)\cos\left(t^2(3t-2)\right) & t^2\cos\left(t^2(3t-2)\right) \\ (3t-2) & t^2+1 \end{bmatrix} \begin{bmatrix} 2t \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2t(3t-2)\cos\left(t^2(3t-2)\right) + 3t^2\cos\left(t^2(3t-2)\right) \\ 2t(3t-2) + 3(t^2+1) \end{bmatrix}.
\end{aligned}$$

And

$$\left.\frac{dF}{dt}\right|_{t=\frac{2}{3}} = D(F\circ\mathbf{c})\left(\frac{2}{3}\right) = \begin{bmatrix} \frac{4}{3} \\ \frac{13}{3} \end{bmatrix}.$$

The other way is to consider $(F\circ\mathbf{c})$ as just another differentiable path, and differentiate with respect to t component-wise.

Question 2. [15 points] Let $\vec{a}$, $\vec{b}$, $\vec{c} \in \mathbb{R}^3$.

(a) If $\vec{a} + \vec{b} + \vec{c} = 0$, show $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$.

(b) If $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$, describe any and all geometric relationships between the three vectors.

Solutions:

(a) **This fact can certainly be established coordinate-wise. However, it is easiest to see that since $\vec{a} + \vec{b} + \vec{c} = 0$, it follows that $\vec{c} = -\vec{a} - \vec{b}$. Thus**

$$\vec{b} \times \vec{c} = \vec{b} \times \left(-\vec{a} - \vec{b}\right) = \vec{b} \times (-\vec{a}) + \vec{b} \times \left(-\vec{b}\right) = -\left(\vec{b} \times \vec{a}\right) + \vec{0} = \left(\vec{a} \times \vec{b}\right)$$

by the various properties of the cross product.

(b) **First, this statement will be true if at least one vector is the zero-vector. Aside from that, the statement will be true if $\exists k \neq 0$ such that $\vec{a} = k\vec{b}$. Then the cross product is the zero-vector. Aside from that, the vector $\vec{a} \times \vec{b}$ may be orthogonal to $\vec{c}$. This would happen if $\vec{c}$ was part of the plane defined by $\vec{a}$ and $\vec{b}$. That is, $\exists k, \ell$ not both 0 (since we assume now that $\vec{c} \neq \vec{0}$) such that $\vec{c} = k\vec{a} + \ell\vec{b}$.**

Question 3. [20 points] For $F(x, y, z) = y^3 - xz - x$ a real-valued function of three variables and $p = (1, 2, 7)$ a point in $\mathbb{R}^3$, do the following:

(a) Evaluate ∇F.

(b) Calculate the directional derivative of F at p in the direction $\mathbf{v} = (-2, 1, -2)$.

(c) Find the equation for the tangent plane to the zero-level set of F at p.

(d) Find a point on the zero-level set of F where the tangent plane is parallel to the xy-plane.

Solutions:

(a) $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(-z-1, 3y^2, -x\right)$.

(b) $D_v F(p) = \nabla F(p) \cdot \frac{\mathbf{v}}{||\mathbf{v}||} = \left(-(7)-1, 3(2)^2, -(1)\right) \cdot \left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{16+12+2}{3} =$ 10.

(c) The equation of the tangent plane to F at p is

$\nabla F(p) \cdot (x-1, y-2, z-7) = (-8, 12, -1) \cdot (x-1, y-2, z-7) = 0.$

Thus the equation is

$$-8(x-1) + 12(y-2) - (z-7) = 0 \textbf{ or } -8x + 12y - z = 9.$$

(d) For the tangent plane to any level set of F to be parallel to the xy-plane, ∇F would have to be "vertical", or $(0, 0, r)$ for some $r \neq 0$. This means that $y = 0$, $z = -1$, and $x = \pm r \neq 0$ given the answer in (a). So on the zero-level set, we get the equation

$$F(x, 0, -1) = 0$$

which we need to solve for a non-zero x. But this equation is solved for ANY x, since

$$F(x, 0, -1) = 0^3 - x(-1) - x = x - x = 0.$$

Hence choose any point $(x, 0, -1)$ where $x \neq 0$ (this ensures that the normal vector is non-zero) and this point is on the zero-level set and the tangent plane is "horizontal". Two good follow up questions: 1) what happens at the point $(0, 0, -1)$ and 2) what does the zero-level set look like near any point on the line $(x, 0, -1)$?

Question 4. [25 points] Suppose that $w = g\left(\frac{x}{y}, \frac{z}{y}\right)$ is a differentiable function of $u = \frac{x}{y}$ and $v = \frac{z}{y}$. Show that for $\vec{x} = (x, y, z) \in \mathbb{R}^3$, the operator $\vec{x} \cdot \nabla$ vanishes on w. That is, show that

$$\vec{x} \cdot \nabla(w) = x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0.$$

Solution: By the Chain Rule, we get immediately that

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial g}{\partial u}\left(\frac{1}{y}\right) \\
\frac{\partial w}{\partial z} &= \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial z} = \frac{\partial g}{\partial v}\left(\frac{1}{y}\right) \\
\frac{\partial w}{\partial y} &= \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial g}{\partial u}\left(-\frac{x}{y^2}\right) + \frac{\partial g}{\partial v}\left(-\frac{z}{y^2}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\vec{x} \cdot \nabla(w) &= x\left(\frac{\partial g}{\partial u}\right)\left(\frac{1}{y}\right) + y\left[\frac{\partial g}{\partial u}\left(-\frac{x}{y^2}\right) + \frac{\partial g}{\partial v}\left(-\frac{z}{y^2}\right)\right] + z\left(\frac{\partial g}{\partial v}\right)\left(\frac{1}{y}\right) \\
&= \left(\frac{x}{y} - \frac{x}{y}\right)\left(\frac{\partial g}{\partial u}\right) + \left(\frac{z}{y} - \frac{z}{y}\right)\left(\frac{\partial g}{\partial v}\right) \\
&= 0.
\end{aligned}$$

Question 5. [15 points] A particle moves in $\mathbb{R}^3$ so that its acceleration is a constant $-\mathbf{k}$. If the particle's initial position at $t = 0$ is $(-1, 0, 2)$ and its velocity at $t = 0$ is the vector $\mathbf{i} + \mathbf{j}$.

(a) When does the particle hit the $z = 0$ plane (the floor?)

(b) Where does it hit the floor?

(c) Express the distance the particle travels between $t = 0$ and the moment it hits the floor as an integral of time alone (you do not need to solve the integral).

Solutions: The acceleration vector is simply the second derivative of the displacement vector, which in this case is the path $\mathbf{c} : [0, \infty) \to \mathbb{R}^3$. Hence $\mathbf{a} = \mathbf{a}(t) = (0, 0, -1) = \mathbf{c}''(t)$. To find c, simply find the second antiderivatives of each of the components separately (this is just Calculus I stuff), and use the initial data to find the constants. Thus $\mathbf{v}(t) = (R_1, R_2, -t + R_3)$ for R_1,R_2, R_3 the three constants of integration. And since $\mathbf{v}(0) = (1, 1, 0)$, we get that $R_1 = R_2 = 1$ and $R_3 = 0$, so that $\mathbf{v}(t) = (1, 1, -t)$. Do this again to get $\mathbf{c}(t) = \left(t - 1, t, -\frac{t^2}{2} + 2\right)$.

(a) Simply solve for the z-component of c to be zero. Hence $0 = -\frac{t^2}{2} + 2$, which is solved for $t = 2$.

(b) $\mathbf{c}(2) = (1, 2, 0)$.

(c) The arclength of c from $t = 0$ to $t = 2$, is

$$\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt = \int_0^2 \sqrt{2 + t^2}\, dt.$$