SPRING 2008: 110.211 HONORS MULTIVARIABLE CALCULUS

Solution to Final Exam Problem number 7

Question 7. [30 points] Use Green's Theorem to prove the following special case of the Change of Variables Formula

$$\iint_D dx\,dy = \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

for the transformation $(u,v) \mapsto (x(u,v), y(u,v))$**.**

Proof. We already know from the second midterm that for D and its boundary ∂D to be oriented compatibly,

$$\iint_D dx\,dy = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy.$$

Now since $x = x(u,v)$ and $y = y(u,v)$, along the boundary we can view this as a reparameterization, and should the direction of travel along ∂D be preserved by the change of parameters, we get

$$\iint_D dx\,dy = \frac{1}{2}\oint_{\partial D^*} -y(u,v)\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) + x(u,v)\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right) \tag{0.1}$$

$$= \frac{1}{2}\oint_{\partial D^*}\left(-y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u}\right)du + \left(-y\frac{\partial x}{\partial v} + x\frac{\partial y}{\partial v}\right)dv \tag{0.2}$$

by the Chain Rule. Now apply Green's Theorem a second time to get

$$\begin{aligned}
\iint_D dx\,dy &= \frac{1}{2}\iint_{D^*}\left[\frac{\partial}{\partial u}\left(-y\frac{\partial x}{\partial v} + x\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left(-y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u}\right)\right]du\,dv \\
&= \frac{1}{2}\iint_{D^*}\left[\left(-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - y\frac{\partial^2 x}{\partial u\partial v} + \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + x\frac{\partial^2 y}{\partial u\partial v}\right) - \left(-\frac{\partial y}{\partial v}\frac{\partial x}{\partial u} - y\frac{\partial^2 x}{\partial v\partial u} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} + x\frac{\partial^2 y}{\partial v\partial u}\right)\right]du\,dv \\
&= \frac{1}{2}\iint_{D^*}\left(2\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - 2\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)du\,dv \\
&= \iint_{D^*}\frac{\partial(x,y)}{\partial(u,v)}du\,dv = \iint_{D^*}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv
\end{aligned}$$

with a final equality valid because the Jacobian here will be positive since if the coordinate transformation preserves direction on the boundary, it preserves orientation on the interior also.

In the case where along the boundary the switch from (x,y) to (u,v) reverses direction, the line integral in Equation 0.1 will change sign. This sign change will filter through to the end, resulting in

$$\iint_D dx\,dy = \iint_{D^*} -\frac{\partial(x,y)}{\partial(u,v)}du\,dv.$$

However, in this case, the transformation is orientation reversing on the interior also, and the Jacobian will be negative. Hence in this case also, we get

$$\iint_D dx\,dy = \iint_{D^*} -\frac{\partial(x,y)}{\partial(u,v)}du\,dv = \iint_{D^*}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

and we are done. $\square$

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