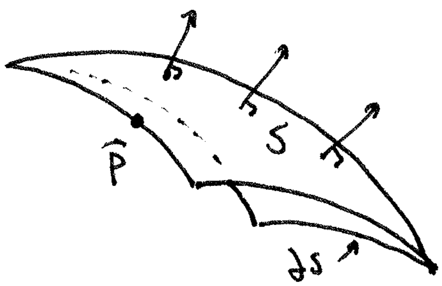
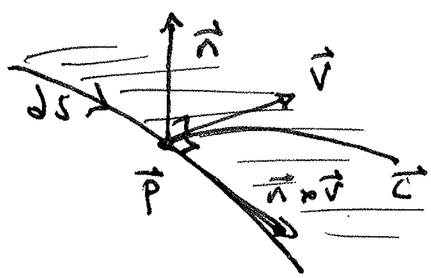
V Orienting a surface automatically orients boundary curves on that surface.



Let S be an oriented, bounded surface in $\mathbb{R}^3$ $\ni$ ∂S is a piecewise C^1 closed curve.

For $\vec{p} = (s_0, t_0) = (x(s_0,t_0), y(s_0,t_0), z(s_0,t_0))$ a pt in ∂S, choose a smooth curve $\vec{c}: [a,b] \to S \subset \mathbb{R}^3$ $\ni$ $\vec{c}(a) = \vec{p}$ and $\vec{c} \cap \partial S = \vec{p}$, call both

$$\vec{n}(\vec{p}) = \lim_{t \to a} \vec{n}(\vec{c}(t)), \text{ and } \vec{v}(a) = \lim_{t \to a} \vec{v}'(t)$$



Here, ~~both~~ $\vec{n}$ and $\vec{v}$ @ $\vec{p}$ are perpendicular. $\vec{n}$ determines a 2-d linear subspace of $\mathbb{R}^3$ containing $\vec{v}$. There exists a ! vector perpendicular to both $\vec{n}$ and $\vec{v}$ in this subspace. This vector is $\vec{n} \times \vec{v}$ and using the RHR determines a direction on ∂S.

This direction is the one used in Green's Thm.!

Thm (Stokes) Let S be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$, where ∂S consists of finitely many piecewise smooth closed curves oriented compatibly. For $\vec{F}$ a C^1-vector field on a domain containing S,

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

(the surface ~~area~~ integral of the curl of $\vec{F}$ on S equals the line integral of $\vec{F}$ along the boundary of S).

Observations

(I) This is sort of like Green's Thm.

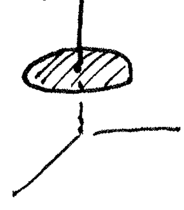
i) LHS measures normal component of curl (amount of twisting in direction through S).

ii) RHS measures circulation of $\vec{F}$ along ∂S (boost or hindrance of $\vec{F}$ on particle moving along ∂S).

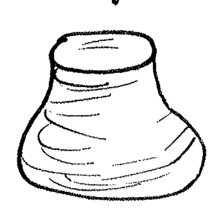
(II) This says alot about the surface!

Let $\partial S = \{(x,y,1) \in \mathbb{R}^3 \mid x^2+y^2=1\}$, $\vec{F}$ C^1 on $\mathbb{R}^3$

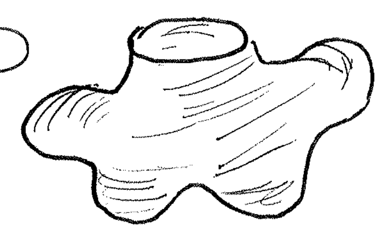
(a)



(b)



(c)

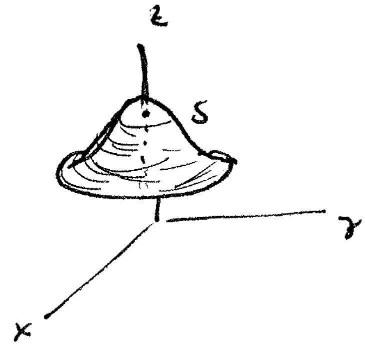


(II) cont'd. In all of these

$$\iint \nabla \times F \bullet d\vec{S} \text{ is the same by } \underline{\text{Stokes Thm.}}$$

__Consequence__ If LHS and RHS are hard to calculate, simply change the interior of S to something easier.

ex. 2 pg 492.



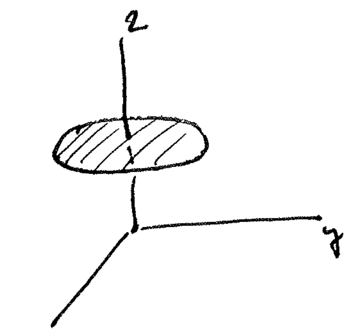
$$\vec{F} = (e^{y+z} - 2y)\vec{i} + (x e^{y+z} + y)\vec{j} + e^{x+y}\vec{k}$$

and $S = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq \frac{1}{e},\ z = e^{-(x^2+y^2)}\}$.

Here, both sides of Stokes's would be very hard to integrate (see book).

However, $\partial S = \{(x,y,\frac{1}{e}) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$.

Choose a new surface $\hat{S} = \{(x,y,\frac{1}{e}) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1\}$.



By Stokes's $\iint_S \nabla \times \vec{F} \bullet d\vec{S} = \iint_{\hat{S}} \nabla \times \vec{F} \bullet d\vec{S}$

$$= \iint_{\hat{S}} \underbrace{(\nabla \times \vec{F} \bullet \vec{n})}_{\text{curl of } \vec{F} \text{ in vertical dir.}} dS$$

$$\nabla \times \vec{F} = (e^{x+y} - x e^{y+z})\vec{i} + (e^{y+z} - e^{x+y})\vec{j} + 2\vec{k}$$

$$= \iint_{\hat{S}} 2\, dS = 2 \cdot (\text{area of } \hat{S}) = \boxed{2\pi}$$

(III) A surface is called <u>compact</u> if it is closed as a set and bounded. It is called <u>closed</u> if it is compact without boundary

By Stokes Thm, the curl of <u>any</u> vector field over a closed surface is 0.

(IV) Let $\vec{F}$ be a <u>conservative</u> <u>vector field</u>: $\vec{F} = \nabla f$ for some potential f, and S any surface which satisfies Stokes thm.

$$\Longrightarrow \oint_{\partial S} \vec{F} \cdot d\vec{s} = 0 \quad \left(\begin{array}{l}\text{The circulation of } \vec{F} \\ \text{along } \partial S \text{ is } 0\end{array}\right)$$

Why? For any C^1 function f, $\nabla \times \nabla f = 0$

Thm (Gauss) Let D be a solid region in $\mathbb{R}^3$ with ∂D a finite set of piecewise smooth, closed, orientable surfaces, oriented w/ $\vec{n}$ pointing away from D. For $\vec{F}$ a C^1 vector field defined on a domain including D, we have

$$\left(\oiint_{\partial D} (\vec{F} \cdot \vec{n})\, dS = \right) \oiint_{\partial D} \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F}\, dV \left(= \iiint_D (\operatorname{div} \vec{F})\, \underset{dx\,dy\,dz}{dV} \right)$$

Note: In $\mathbb{R}^2$, any compact ~~closed~~ ~~bounded~~ domain with non-empty interior has a ~~closed~~ set of closed curves as boundary

In $\mathbb{R}^3$, any compact solid region w/ non empty interior has a set of closed surfaces as boundary.

The proof is straightforward:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$. Then

(1) $$\iiint_D \operatorname{div} F \, dV = \iiint_D \frac{\partial F_1}{\partial x} dV + \iiint \frac{\partial F_2}{\partial y} dV + \iiint \frac{\partial F_3}{\partial z} dV$$

and

(2) $$\iint_{\partial D} \vec{F} \cdot \vec{n} \, dS = \iint_{\partial D} F_1\vec{i} \cdot \vec{n} \, dS + \iint_{\partial D} F_2\vec{j} \cdot \vec{n} \, dS + \iint_{\partial D} F_3\vec{k} \cdot \vec{n} \, dS$$

Show each summand in (1) is respectively equal to each in (2).

exercise: Do this when D is elementary in all 3 directions.

Interpretations

(I) What is divergence? Intuition: measures the infinitessimal expansion of volume under the flow of a vector field.

Actual: It measures the aggregate flow across the boundary of an infinitessimal ~~sphere~~ ball centered at a pt.

Thm Let $\vec{F}$ be a C^1 vector field in some nbhd of $\vec{p} \in \mathbb{R}^3$. For S_a the 2-sphere of radius centered @ $\vec{p}$ and oriented w/ outward normal,

$$\operatorname{div} \vec{F}(\vec{p}) = \lim_{a \to 0^+} \frac{3}{4\pi a^3} \oiint_{S_a} \vec{F} \cdot d\vec{S}$$

proof. For any $f \in C^0[D \subset \mathbb{R}^3, \mathbb{R}]$, D a bounded solid region, $\exists \, q \in \mathbb{R}^3 \ni \iiint_D f(x,y,z)\, dV = f(q) \cdot \text{volume}(D)$.

(MVT for triple integrals).

proof cont'd.

Given $\vec{p}$ and S_a as in thm, $\exists\, \vec{q} \in B_a(\vec{p})$

$$\ni \iiint_{B_a} (\operatorname{div} \vec{F})\, dV = \operatorname{div}(\vec{F}(\vec{q})) \cdot (\text{volume } B_a)$$

$$= \tfrac{4}{3}\pi a^3 \cdot \operatorname{div} \vec{F}(\vec{q})$$

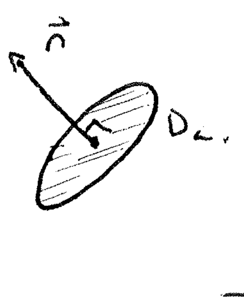
Now $$\lim_{a \to 0^+} \frac{3}{4\pi a^3} \oiint_{S_a} \vec{F} \cdot d\vec{S} \overset{\text{Gauss thm}}{=\!=} \lim_{a \to 0^+} \frac{3}{4\pi a^3} \iiint_{B_a} \operatorname{div} \vec{F}\, dV$$

$$= \lim_{a \to 0^+} \frac{3}{4\pi a^3} \left(\tfrac{4}{3}\pi a^3 \operatorname{div} \vec{F}(\vec{q}) \right)$$

$$= \operatorname{div} \vec{F}(\vec{p}).$$

Gauss' Thm – The amount of volume created or lost upon flowing along a vector field in D equals to total amount flowing across the boundary ∂D.

(II) What is curl? Let $\vec{F}$ be a C^1 vector field in a nbhd of $\vec{p} \in \mathbb{R}^3$. Choose a unit vector $\vec{n}$ and a disk D_a of radius a centered @ $\vec{p}$ and normal to $\vec{n}$. Call $C_a = \partial D_a$ and orient C_a and D_a compatibly to $\vec{n}$.



$\Rightarrow$ The component of curl$(\vec{F})$ in the dir of $\vec{n}$ is

$$\operatorname{curl} \vec{F}(\vec{p}) \cdot \vec{n} = \lim_{a \to 0^+} \frac{1}{\pi a^2} \oint_{C_a} \vec{F} \cdot d\vec{s}$$

Curl is the infinitesimal circulation of $\vec{F}$ along a loop perpendicular to the direction ~~[illegible]~~ of flow.

pf. almost exactly like previous proof using Stokes Thm.

Notes ① $\operatorname{div}\vec{F}(\vec{p})$ is also called the flux density of $\vec{F}$ @ $\vec{p}$: limit of the flux per unit volume @ $\vec{p}$ of $\vec{F}$.

② $\operatorname{curl}\vec{F}(\vec{p})$ is also called the circulation density of $\vec{F}$ @ $\vec{p}$: the limit of the circulation per unit area of $\vec{F}$ @ $\vec{a}$.

Stokes' Thm - the total rotational effect of a vector field on a surface ~~can be~~ is equal to the aggregate push or hindrance of a particle on the edge.