EXTRA STUFF: A PRODUCT RULE FOR LIMITS WITH VECTOR-VALUED FUNCTIONS

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I mentioned in class that the notion of a limit of a function of more than one-independent variable can be tricky to study due to the problem of being able to "approach" a particular point $\mathbf{a} \in \mathbb{R}^n$ from so many ways when $n > 1$. However, I also talked about how limits, even when the domains and co-domains are bigger than $\mathbb{R}$, still behave much like their counter parts for scalar-valued functions of one independent variable (the Calculus I-II sort). For instance, the limit of a sum of functions at a point equals the sum of the limits of the summand functions at the point, provided each of the summand limits exists, that is. Limits of multiples of functions equals the multiple of the limit of a function also. And the limit of a product of functions is the product of the limits of the factor functions, again when they each exist. The caveat in this last case, though, is that the product of two vector-valued functions must actually make sense for this statement to make sense at all. So, under teh implicit idea that the product actually makes sense in this case, the Product Rule for vector-valued functions would in fact work. Let's look at some examples:

First, the book claims the scalar-valued function version of a product rule:

Theorem (Product Rule for Functions on $\mathbb{R}^n$)**.** *For $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, let* $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ *and* $\lim\limits_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$ *exist. Then*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})g(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right).$$

You can see how this may be a problem if the chosen functions above were vector-valued. Just how does one multiply together the output values of the two functions, given that they are vectors? The answer is that there are ways to multiply vectors together. Many, in fact. Does the Product Rule hold if we allow for such multiplications? In fact, it does:

Claim. *For $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^{m_2}$, let* $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$ *and* $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})$ *exist. Then*

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \cdot \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})$$

when the products on each side are the same and when they make sense.

Proving this statement for ANY possible product is not necessary here. But we can show that it holds for the common products we use in this course. Here we show the rule for the dot product. To start, recall Theorem 2.6 in the text, which states that for $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{m_1}$, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))^T$, the limit $\lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \in \mathbb{R}^m$, where $\mathbf{L} = (L_1, \ldots, L_m)^T$, if for $i = 1, \ldots, m$, we have $\lim\limits_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = L_i$. Here, the limit of a vector-valued function will exist iff each of its component scalar-valued functions has a limit, and that limit will be the vector of the component limits.

Proof when $m_1 = m_2$ using the dot product. . Recall the dot product of two $\mathbb{R}^n$-vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ is

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Now for $\mathbf{f}$ and $\mathbf{g}$ as above, we can use the dot product to write their product as a scalar-valued functions

$$h : \mathbb{R}^n \to \mathbb{R}, \quad h(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) g_i(\mathbf{x}).$$

Now assuming that both $\lim\limits_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x})$ and $\lim\limits_{\mathbf{x} \to \mathbf{a}} \mathbf{g}(\mathbf{x})$ exist, means that each of the limits of the component functions of each exist at $\mathbf{a}$. But this means

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \lim_{\mathbf{x} \to \mathbf{a}} \sum_{i=1}^{n} f_i(\mathbf{x}) g_i(\mathbf{x})$$

$$= \sum_{i=1}^{n} \lim_{\mathbf{x} \to \mathbf{a}} f_i(\mathbf{x}) g_i(\mathbf{x}) = \sum_{i=1}^{n} \left(\lim_{\mathbf{x} \to \mathbf{a}} f_i(\mathbf{x}) \right) \left(\lim_{\mathbf{x} \to \mathbf{a}} g_i(\mathbf{x}) \right)$$

due to the product and sum rules for limits of scalar-valued functions. But notice that

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) \cdot \lim_{\mathbf{x} \to \mathbf{a}} \mathbf{g}(\mathbf{x}) = \lim_{\mathbf{x} \to \mathbf{a}} (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))^T \cdot \lim_{\mathbf{x} \to \mathbf{a}} (g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))^T$$

$$= \left(\lim_{\mathbf{x} \to \mathbf{a}} f_1(\mathbf{x}), \ldots, \lim_{\mathbf{x} \to \mathbf{a}} f_m(\mathbf{x}) \right)^T \cdot \left(\lim_{\mathbf{x} \to \mathbf{a}} g_1(\mathbf{x}), \ldots, \lim_{\mathbf{x} \to \mathbf{a}} g_m(\mathbf{x}) \right)^T$$

$$= \sum_{i=1}^{n} \left(\lim_{\mathbf{x} \to \mathbf{a}} f_i(\mathbf{x}) \right) \left(\lim_{\mathbf{x} \to \mathbf{a}} g_i(\mathbf{x}) \right).$$

As they are equal, we are done.   □

Exercise 1. *Do this construction for the case where $m_1 = m_2 = 3$ using the cross product.*

Question 1. *For a product of vectors to make sense, must it be the case that $m_1 = m_2$?*

Exercise 2. *Do this when $m_1 = 1$ and $m_2 > 1$.*

Now create a matrix-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{p \times m}$ defined by

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \cdots & f_{1m}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \cdots & f_{2m}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{p1}(\mathbf{x}) & f_{p2}(\mathbf{x}) & \cdots & f_{pm}(\mathbf{x}) \end{bmatrix}.$$

Really the co-domain here can be thought of as basically the real space of dimension pm (think of a 2×3-matrix as having 6 elements, just like a 6-vector). Assume that the limit of a function like $\mathbf{f}$ will exist at a point $\mathbf{a} \in \mathbb{R}^n$ iff ALL of the component functions have limits as scalar-valued functions, and the limit $\mathbf{L}$ will be an $p \times m$ matrix with elements L_{ij}.

Exercise 3. *Establish the Product Rule for $\mathbf{f}$ defined this way and for $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ defined as at the top of this essay.*