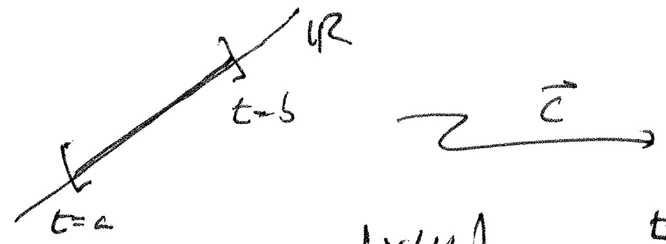
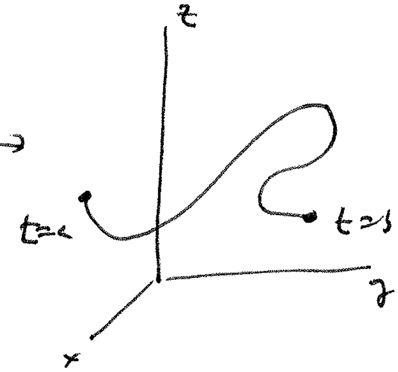
Parametrized Surfaces

We parametrize a curve in $\mathbb{R}^n$ via a map:

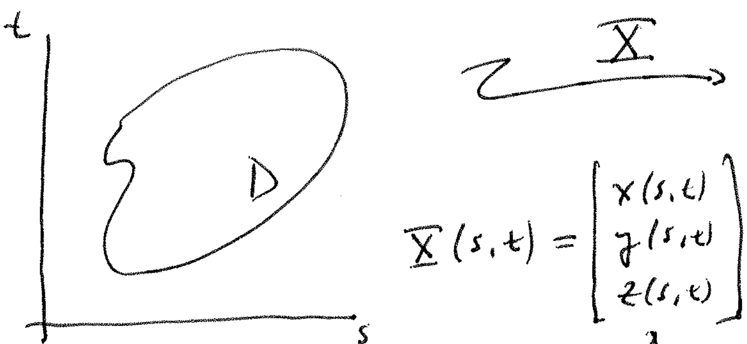


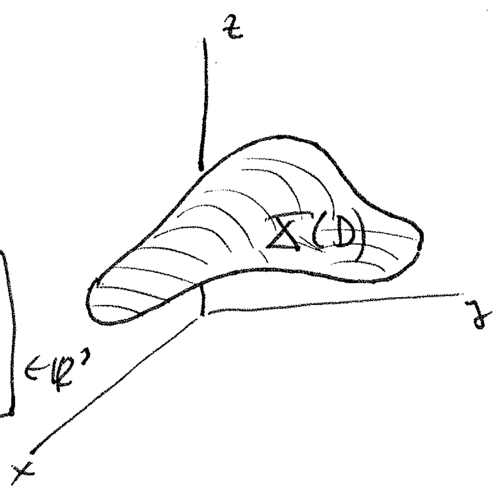


$$\vec{c}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \in \mathbb{R}^3.$$

parametric eqn

Let $D \subset \mathbb{R}^2$ be a connected open set along with some or all of its bdy pts. A parametrized surface in $\mathbb{R}^3$ is a C^0 function $\underline{X}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ that is 1-1 on interior of D

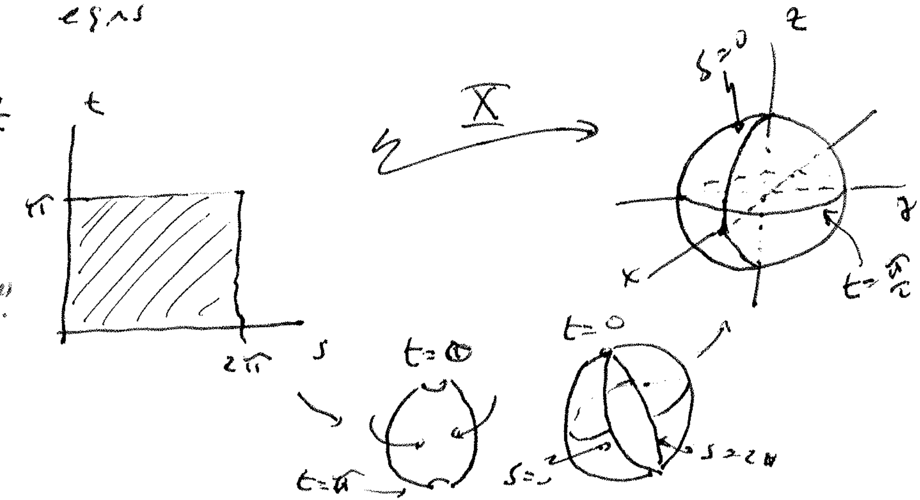




$$\underline{X}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} \in \mathbb{R}^3$$

parametric eqns

ex. $x = a \cos s \sin t$
$y = a \sin s \sin t$
$z = a \cos t$



Here coordinate constant coord lines curves are longitude and latitude lines on S^2.

Example 5 is beautiful! The construction of π?

Def. A surface $S = \underline{X}(D)$ is called diff. if coord functions are diff, and

$$\underline{X}_s(s_0,t_0) = \frac{\partial \underline{X}}{\partial s}(s_0,t_0) = \left(\frac{\partial x}{\partial s}\bigg|_{(s_0,t_0)}, \frac{\partial y}{\partial s}\bigg|_{(s_0,t_0)}, \frac{\partial z}{\partial s}\bigg|_{(s_0,t_0)} \right)$$

same for $\underline{X}_t(s_0,t_0)$.

Here $\underline{X}_s$ and $\underline{X}_t$ are always tangent to the surface and $\underline{X}_s \times \underline{X}_t$ is always normal to the surface. (if it is nonzero and C^1). Call $N(s_0,t_0) = \underline{X}_s \times \underline{X}_t(s_0,t_0)$.

Def $S = \underline{X}(D)$ is called <u>smooth</u> @ $\underline{X}(s_0,t_0)$ if $\underline{X}$ is C^1 in a nbhd of (s_0,t_0) and if $N(s_0,t_0) \neq \vec{0}$.
S is smooth if it is smooth everywhere.

Note: C^1 ensures no edges only if $N(s_0,t_0) \neq \vec{0}$, like for curves!

ex. For $\underline{X} = S^2$, $\underline{X}_s = (-a\sin s\sin t, a\cos s\sin t, 0) = (-y, x, 0)$

$\underline{X}_t = (a\cos s\cos t, a\sin s\cos t, -a\sin t)$

$N = \underline{X}_s \times \underline{X}_t = -a\sin t\,(x,y,z)$ <u>Do this</u>.

Here N is nonzero except at $t=0$ and π (the poles).

S^2 is smooth @ poles, but not according to the parametrization.

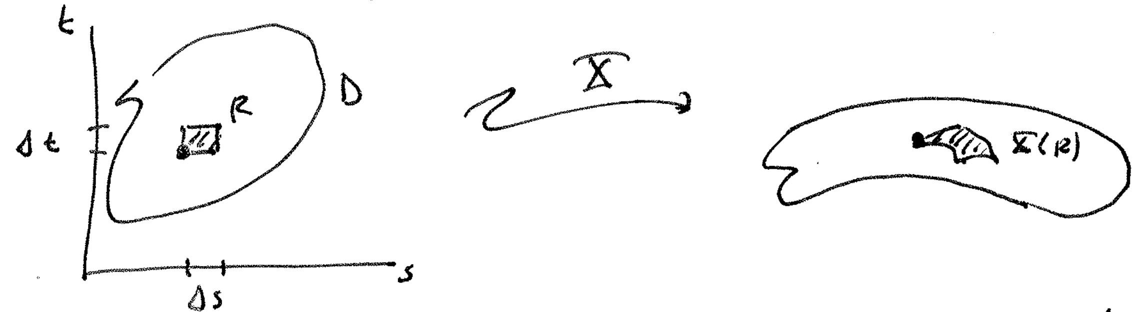
"You cannot walk east or west @ the north pole, you can only walk south!".

Def A piecewise smooth param. surface is the union of images of finitely many param. surfaces $\mathbf{X}_i : D_i \to \mathbb{R}^3$, where each D_i is (1) elementary, (2) C^1 except possibly along ∂D_i, and (3) each $S_i = \mathbf{X}_i(D_i)$ is smooth except at a possible finite # of pts.

Area of a parameterized surface.

Recall length of $\vec{c}:[a,b] \to \mathbb{R}^n$ is $\int_a^b \|c'(t)\|\,dt$ and was independent of parametrization.

In 2-dim.



Here $\mathbf{X}(R)$ need not be a rectangle, but can be approximated by one with sides $\mathbf{X}_s(s_0,t_0)\Delta s$, $\mathbf{X}_t(s_0,t_0)\Delta t$, so

$$\text{Area}\,(\mathbf{X}(R)) \approx \|\mathbf{X}_s(s_0,t_0)\Delta s \times \mathbf{X}_t(s_0,t_0)\Delta t\|$$
$$= \|\mathbf{X}_s(s_0,t_0) \times \mathbf{X}_t(s_0,t_0)\|\,\Delta s\,\Delta t$$

(This is the area of the unit square in tangent plane to $\mathbf{X}(D)$ @ (s_0,t_0) scaled by Δs & Δt.

In the limit, as $\Delta s, \Delta t \to 0$, $\longrightarrow \|\mathbf{X}_s(s_0,t_0) \times \mathbf{X}_t(s_0,t_0)\|\,ds\,dt$

and since $\text{area}(D) = \iint_D dA$, we set

$$\text{area}(S) = \iint_D \|\mathbf{X}_s \times \mathbf{X}_t\|\,ds\,dt = \iint_D \|N(s,t)\|\,ds\,dt$$
$$= \iint dS$$

where $dS = \|N(s,t)\| \, dA$ is the area form on the surface.

Notes: ① This $dS = \|N(s,t)\| \, dA$ is the analogue to the expression $ds = \|\vec{c}\,'(t)\| \, dt$ in the path integral.

② For $\Phi(s,t) = (x(s,t), y(s,t), z(s,t)) \in \mathbb{R}^3$,

$$\Phi_s \times \Phi_t = \left(\frac{\partial(y,z)}{\partial(s,t)}, \; -\frac{\partial(x,z)}{\partial(s,t)}, \; \frac{\partial(x,y)}{\partial(s,t)} \right) \quad \text{so}$$

that

Jacobians

$$\text{area}(S) = \iint_D \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2} \, ds \, dt$$

Compare this to arclength in Calc I: $\vec{c} = (x(t), y(t))$

$$\text{arclength}(\vec{c}) = \int_{\vec{c}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

<u>Integrating a function on a parameterized surface.</u>

- Like for path integrals, want to parametrize the surface and use the parameters as variables of integration, but integral <u>must</u> be parameter-independent.
- In theory, would look like $\iint_{\text{surface}} f \, dS$

where $dS = \|N(s,t)\| \, ds \, dt$

<u>Set up</u>: Let $\mathbf{X}: D \to \mathbb{R}^3$ be a smooth param. surface, where $D \subseteq \mathbb{R}^2$ is bounded. Let f: be C^0 on a domain that includes $\mathbf{X}(D)$.

Then the surface integral of f along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} f \, dS = \iint_D f(\mathbf{X}(s,t)) \, \|\mathbf{X}_s \times \mathbf{X}_t\| \, ds\, dt$$

$$= \iint_D f(x(s,t), y(s,t), z(s,t)) \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \cdots + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2} \, ds\, dt$$

This is the integral of a scalar-valued fn over $\mathbf{X}(D) \subset \mathbb{R}^3$. In practice, everything can be written out in terms of s, t and treated as a standard double integral.

<u>Def</u>. Let $\mathbf{X}: D \to \mathbb{R}^3$ be a smooth parametrized surface with $D \subset \mathbb{R}^2$ bounded. Let $\vec{F}$ be a C^0 vector field on a domain including $\mathbf{X}(D)$. The vector surface integral of $\vec{F}$ along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\mathbf{X}(s,t)) \cdot N(s,t) \, ds\, dt$$

$$= \iint_D \vec{F}(x(s,t), y(s,t), z(s,t)) \cdot \underbrace{\left(\frac{\partial(y,z)}{\partial(s,t)}, -\frac{\partial(x,z)}{\partial(s,t)}, \frac{\partial(x,y)}{\partial(s,t)}\right) ds\, dt}_{d\vec{S}}$$

Here $d\vec{S} = N(s,t)\, ds\, dt$

Let $\vec{n}(s,t) = \dfrac{N(s,t)}{\|N(s,t)\|}$ be the unit normal to $\Sigma(D)$.

$$\Longrightarrow \iint_{\Sigma} \vec{F} \cdot d\vec{S} = \iint_{D} \vec{F} \cdot \vec{n}\, dS$$

Compare this to the vector line integral $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} (\vec{F} \cdot \vec{T})\, ds$.

Interpretation

- Since this integral is a dot product with the normal to the surface $N(s,t) = \Sigma_s(s,t) \times \Sigma_t(s,t)$,

  $\iint_{\Sigma} \vec{F} \cdot d\vec{S}$ measures the vector field flow through Σ and is called the flux of $\vec{F}$ through $\Sigma(D)$.

- Compare to $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ measured the vector field flow along $\vec{c}$ and is called the circulation of $\vec{F}$ along $\vec{c}$.

Other Interpretations

(I) Given a parametrization $\Sigma : D_1 \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ and a C^0, 1-1, onto fnc $\vec{H} : D_2 \xrightarrow{\subseteq \mathbb{R}^2} D_1$, call $\Upsilon : D_2 \to \mathbb{R}^3$ a reparametrization of Σ if $\Upsilon(u,v) = \Sigma(H(u,v))$

This reparam. is smooth if both Σ, Υ are and H is C^1.

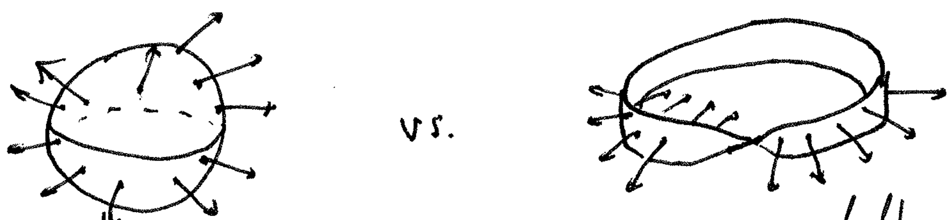
(II) <u>Thm</u> For f C^0 on a domain including a smooth $\mathbf{X}: D \longrightarrow \mathbb{R}^3$, then for any $\mathbf{Y}$ a smooth reparametrization of $\mathbf{X}$,

$$\iint_{\mathbf{X}} f\, dS = \iint_{\mathbf{Y}} f\, dS$$

(III) For a curve, an <u>orientation</u> is a choice of a continuously varying unit tangent vector along $\vec{c}$.

For a surface, an <u>orientation</u> is a choice of a continuously varying unit <u>normal</u> vector to $\mathbf{X}$ (above vs. below, inside vs. outside).

If this is possible, then surface is called orientable.



orientable vs. non-orientable

~~Def~~ Recall $\vec{N}(s,t) = \mathbf{X}_s \times \mathbf{X}_t = -\mathbf{X}_t \times \mathbf{X}_s$

(IV) <u>Thm</u> If a reparametrization $\mathbf{Y}$ preserves orientation (i.e. $\text{Jacobian}(H) \geq 0$ everywhere)

$$\Longrightarrow \iint_{\mathbf{Y}} \vec{F} \cdot d\vec{S} = \iint_{\mathbf{X}} \vec{F} \cdot d\vec{S}$$

otherwise minus.