THE LENGTH OF A PATH IN $\mathbb{R}^n$ IS INDEPENDENT OF ITS PARAMETERIZATION.

110.211 HONORS MULTIVARIABLE CALCULUS
PROFESSOR RICHARD BROWN

Proposition 1. *Show that the length of a differentiable curve in $\mathbb{R}^n$ is independent of its parameterization.*

The easiest way to argue this is simply to say that since any parameterization (we are assuming that parameterizations have non-zero velocity, as in the book) is equivalent to the arc-length re-parameterization, this implies immediately that all such parameterizations are equivalent, in terms of length.

To argue in terms of individual parameterizations is just as straightforward: First, some simplifications to make the actual point of this exercise easier to see. Suppose the actual curve in $\mathbb{R}^n$ has no corners, and that we will deal only with differential paths with non-zero velocity. Also, let's consider only paths that have parameterizations that start from the same end of the curve and that the domain of these paths is a finite, closed interval. Again, these are not restrictions on the validity of the claim. Rather they make the proof more illuminating by clearing clutter.

Proof. Let $\mathbf{c} : I = [a, b] \to \mathbb{R}^n$ and $\mathbf{d} : J = [\alpha, \beta] \to \mathbb{R}^n$ be two differentiable paths, where $\mathbf{c}(I) = \mathbf{d}(J)$ and $\mathbf{c}(a) = \mathbf{d}(\alpha)$. Since these are parameterizations, it follows immediately that $\mathbf{c}(b) = \mathbf{d}(\beta)$ and $\forall s \in J$, $\exists t \in I$ such that $\mathbf{d}(s) = \mathbf{c}(t)$. Thus there is a function $p : J \to I$, $p(s) = t$, so that $\mathbf{d}(s) = \mathbf{c}\left(p(s)\right)$. Since the paths are differentiable with non-zero velocity, this will imply that p is C^1 also, and

$$\mathbf{d}'(s) = \left(\mathbf{c} \circ p\right)'(s) = \mathbf{c}'(p(s))p'(s). \tag{1}$$

Remark 2. Strictly speaking, the inside function of a composite-function need not be differentiable for the composition to be differentiable (think of $f \circ g$, where $f = x^2$ and $g = |x|$ on an open interval of 0), but in this case it will be. The reason is subtle, but involves the fact that we are assuming nonzero velocity of our paths. One can have a differentiable path in $\mathbb{R}^n$ whose image has a corner in it. The trick is to make sure that the velocity at the corner point is zero. Then the "bead on the wire" can smoothly change direction and continue on. For our two differentiable paths, at each point, the velocity vector of one must be a positive multiple of the velocity vector of the other (they ARE the same curve after all). And since both vectors are varying continuously along the curve, this positive multiple must also vary continuously. But this positive multiple is precisely the derivative of p (see Equation 1). Since p' is continuous, p is differentiable here.)

This remark tells us more. It tells us that p is 1-1 onto its image, and is strictly increasing, so that $p' > 0$ on J. To finish the proof, we have

$$\begin{aligned}
\textbf{length of curve} &= \int_\alpha^\beta ||\mathbf{d}'(s)||\,ds \\
&= \int_\alpha^\beta ||\left(\mathbf{c}\circ p\right)'(s)||\,ds \\
&= \int_\alpha^\beta ||\mathbf{c}'(p(s))p'(s)||\,ds \\
&= \int_\alpha^\beta ||\mathbf{c}'(s)||p'(s)\,ds, \qquad \textbf{since } p'(s) \textbf{ is a positive scalar} \\
&= \int_{a=p(\alpha)}^{b=p(\beta)} ||\mathbf{c}'(t)||\,dt, \qquad \textbf{where } t = p(s) \textbf{ and } dt = p'(s)\,ds.
\end{aligned}$$

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