110.108 CALCULUS I

Week 3 Lecture Notes: September 12 - September 16

Lecture 1: Section 2.4 Limits

Today I started with the formal, precise definition of a limit. This definition allows for a precise calculation to show that a limit exists and is a particular number L. There will be times when it is necessary to use this definition, so a good understanding of how it works is necessary for a good understanding of the underlying concepts in ALL of Calculus.

Definition 1. Let $f(x)$ be defined on an open interval containing a point a, except possibly at a. Then we say the limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x\to a} f(x) = L,$$

if, given any choice of $\epsilon > 0$, we can find a $\delta > 0$ such that,

$$\text{If } 0 < |x-a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Take come time to absorb this definition. Here are some comments:

- The ϵ here is a number which expresses how close the function values will be to a number L. It is a number which is given to you. Note that the condition $|f(x) - L| < \epsilon$ can be rewritten
$$-\epsilon < f(x) - L < \epsilon \quad \text{or} \quad L - \epsilon < f(x) < L + \epsilon.$$
This last expression means that function values are restricted to lie in the interval $(L - \epsilon, \mathrm{L} + \epsilon)$.
- the quantity $\delta > 0$ is a number you would construct, and measures how close you must restrict the x-values to be from a to ensure that the function values stay "close enough" (within ϵ). Technically, you want to choose a value for δ so that all function values of all values of x inside the interval $(a - \delta, a + \delta)$, but not at a, lie within ϵ of L.
- Usually, the δ you will need to construct will depend on the size of ϵ. In this way, δ becomes a function of ϵ, and expressing the value of δ in terms of ϵ is necessary to show via this definition that the limit exists and is L.
- This definition does not allow for the calculation of L. unfortunately. It allows you to prove that a limit will be L if the L is correctly chosen, and to prove that an incorrect value for L cannot possibly be the limit. But one must find another way to actually find the correct L. This is where intuition helps much.

In the diagram below at left, the function $f(x)$ seems to have a limit at $x = a$. But given the $\epsilon > 0$ shown, the choice of $\delta > 0$ is NOT a good choice for the given ϵ. Indeed, there are points in the interval $(a - \delta, a + \delta)$, represented by the vertical band, where $f(x)$ is not in the interval $(L - \epsilon, L + \epsilon)$, represented by the horizontal band. The point $x = p$ is one such point. On the right is the same graph but where the choice of $\delta > 0$ is a good one for this $\epsilon > 0$. Notice that the choice of δ will be a good one when the graph of $f(x)$ across the entire δ-interval is in the horizontal band. See if this makes sense to you.

Also, a couple of more points. If, given an $\epsilon > 0$, you find a good choice of $\delta > 0$, then ANY other choice of $\delta > 0$ will also work. Right? And for the limit to exist, one must ba able to work with ANY given $\epsilon > 0$. For functions with gaps, say, large choices of $\epsilon > 0$ will seem to work fine. But if the definition does not hold for all possible $\epsilon > 0$, then the definition does not hold at all. And lastly, this concept is the first "leap" in mathematical understanding from how something behaves near a point, to what we can say is happening AT the point. One can think of a bunch of ϵ's going to 0 here, and forcing us to squeeze the δ-band also to

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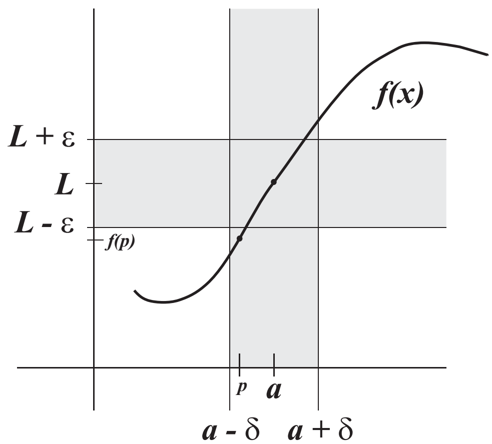
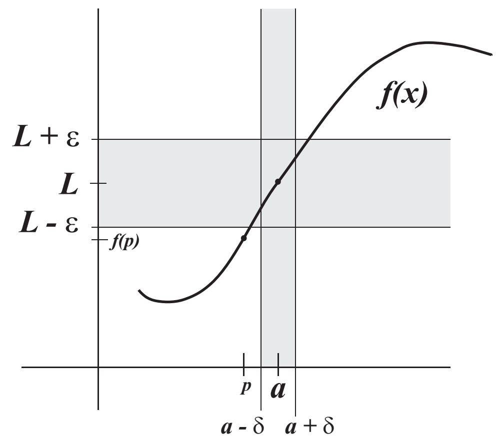
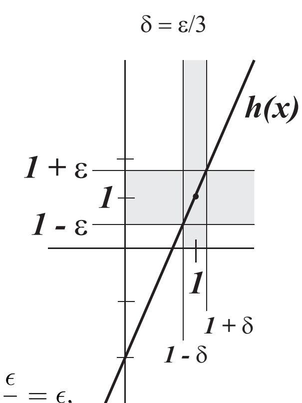
0. We never actually get to 0 in either case, but the result is still concrete: if this construction works for all possible $\epsilon > 0$, then the limit will exist AT the point.





Example 2. *Let* $h(x) = 3x - 2$. *Show that* $\lim_{x \to 1} h(x) = 1$.



To show this, we start with an ϵ as given, which gives us a small open interval centered at $L = 1$ defined by $(1 - \epsilon, 1 + \epsilon)$. This is the horizontal band. Can we find a $\delta > 0$ so that all values of $h(x)$ for x in $(1 - \delta, 1 + \delta)$ are in this horizontal band? The answer is yes, but can we calculate δ as a function of ϵ? To see this, work with $|f(x) - L| < \epsilon$:

$$|f(x) - L| = |(3x - 2) - 1| = |3x - 3| = 3|x - 1| < \epsilon,$$

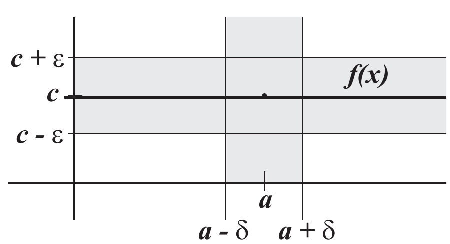
so that $|x - 1| < \frac{\epsilon}{3}$. Since this last inequality is much like the δ-inequality in the definition, we get a formula for δ as a function of ϵ, namely $\delta = \frac{\epsilon}{3}$.

Why does this work? Well, if we choose $\delta = \frac{\epsilon}{3}$, then we get

$$|f(x) - L| = |(3x - 2) - 1| = |3x - 3| = 3|x - 1| < 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon,$$

which is what we wanted.

Example 3. *Show that* $\lim_{x \to a} c = c$.



Let $f(x) = c$. Again, take an $\epsilon > 0$, and form the horizontal band, as in the figure. Here, convince yourself that ANY choice of $\delta > 0$ will work here. Note that for any choice of $\delta > 0$, when $0 < |x - a| < \delta$, then

$$|f(x) - c| = |c - c| = 0 < \epsilon$$

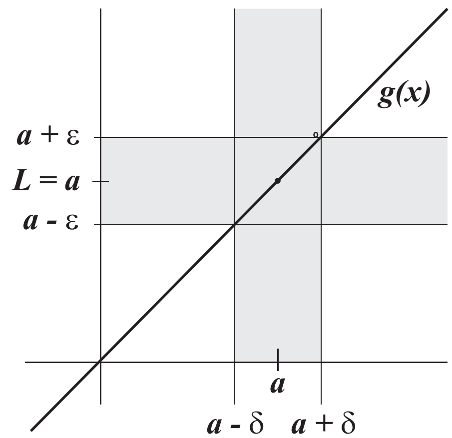
for ANY choice of $\epsilon > 0$. Really, there is not much more to say here.

Example 4. *Show that* $\lim_{x \to a} x = a$.

Let $g(x) = x$. Again, take an $\epsilon > 0$, and form the horizontal band, as in the figure. Here, finding the δ-band so that all values of x inside the δ-band have function values inside the ϵ-band of $L = a$ should be straightforward: Notice the inequality

$$|g(x) - a| = |x - a| < \epsilon$$

is exactly like the δ-inequality in the definition. In fact, setting $\delta = \epsilon$ will do the trick. This make the intersection of the vertical band and the horizontal band a square, and the graph of $g(x)$ is the diagonal.



We are now in a position to state a very useful theorem:

Theorem 5. *Let $f(x)$ be a polynomial or a rational function, and c a point in the interior of the domain of $f(x)$. Then $\lim\limits_{x\to c} f(x) = f(c)$.*

Proof. Let $f(x)$ be a polynomial. Then $f(x)$ is simply a finite sum of multiples of non-negative powers of x. Hence, since we know from above that the limit of a constant is the constant, and the limit of x as x approaches a is a, we can use these facts and the limit laws to show this. Indeed, write $f(x)$ as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

Then we use the Product, Sum and Constant Multiple Laws in the Limit Laws to break down the limit. This is under the understanding that if at some point the limit of a constituent part does not exist, I cannot break down the limits using the limit laws. We will see. We get

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c}\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0\right)\\
&= \lim_{x\to c}\left(a_n x^n\right) + \lim_{x\to c}\left(a_{n-1}x^{n-1}\right) + \ldots + \lim_{x\to c}\left(a_1 x\right) + \lim_{x\to c}\left(a_0\right)\\
&= \left(\lim_{x\to c} a_n\right)\left(\lim_{x\to c} x^n\right) + \left(\lim_{x\to c} a_{n-1}\right)\left(\lim_{x\to c} x^{n-1}\right) + \ldots + \lim_{x\to c}\left(a_0\right)\\
&= \left(\lim_{x\to c} a_n\right)\left(\lim_{x\to c} x\right)^n + \left(\lim_{x\to c} a_{n-1}\right)\left(\lim_{x\to c} x\right)^{n-1} + \ldots + \left(\lim_{x\to c} a_1\right)\left(\lim_{x\to c} x\right) + \lim_{x\to c} a_0\\
&= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0\\
&= f(c).
\end{aligned}$$

Note that there is a step where we use the fact that x^n equals x times itself n-times. And $\lim\limits_{x\to c} x^n = \left(\lim\limits_{x\to c} x\right)^n$. Again, this is just the Product Limit Law, and since $\lim\limits_{x\to c} x$ exists, this works.

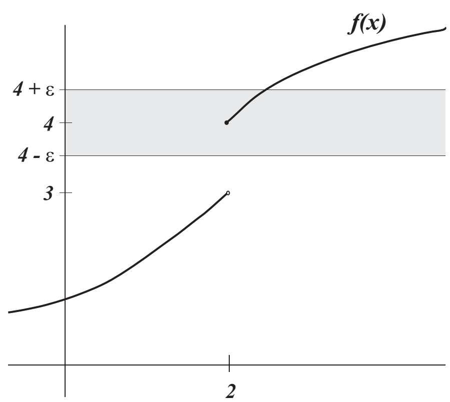
What if $f(x)$ is a rational function. Well, then, $f(x) = \frac{p(x)}{q(x)}$. If c is in the domain of $f(x)$, then the denominator certainly does NOT have a limit of 0 at c. Then we can write

$$\lim_{x\to c} f(x) = \lim_{x\to c}\frac{p(x)}{q(x)} = \frac{\lim_{x\to c} p(x)}{\lim_{x\to c} q(x)} = \frac{p(c)}{q(c)} = f(c),$$

al of which is well-defined using the Quotient Limit Law. We are done. □

It is instructive to see just how a limit could fail to occur. One way is if there is a clear gap in the graph of a function at a point $x = a$. Here, it is easy to see that the function values will "tend" to different values upon approaching a from the two sides. However, how does the definition fail?

Example 6. *Show that* $\lim_{x\to 2} f(x)$ *does not exist for* $f(x)$ *given in the graph.*



The choice of a proper L is unclear here. Let's try $L = 4$. Then, if we choose $\epsilon = \frac{1}{2}$, say, There is no possible interval around $a = 2$ where all functions values on that interval lie inside the horizontal band. All function values, for points in any interval and less than 2 will lie outside the band. hence no such $\delta > 0$ can exist. Hence $L = 4$ cannot be the limit. But by a similar argument, neither can $L = 3$. And if we try to choose $3 < L < 4$, then we can find a very small $\epsilon > 0$ where no function values of $f(x)$ will lie inside the horizontal band defined by $(L - \epsilon, L + \epsilon)$. Hence these choices of L won't work either. interval around Let $f(x) = c$. In this way, one can argue that the limit cannot exist because no number L can satisfy the definition.

One final note: Again, the definition of a limit above is most often used to show that a particular number L is the limit. It typically isn't used to actually calculate the number L. You will have to use Limit Laws or some other means to come up with the number L. The definition is the precise statement allowing you to "prove" your choice is the right one.

Lecture 2: Section 2.4-2.3 Limits