PROOF OF A SPECIAL CASE OF THE IMPLICIT FUNCTION THEOREM

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Here we prove a special case of the Implicit Function Theorem for a C^1 real-valued function on $X \subseteq \mathbb{R}^3$. The generalization to a real valued-function on $\mathbb{R}^n$ is straightforward. The generalization to vector-valued functions is a bit more involved, but similar.

Theorem. *Let $F : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ be of class C^1 and let $\mathbf{a} = (a_1, a_2, a_3)$ be a point of the level set $S = \{(x,y,z) \in \mathbb{R}^3 | F(x,y,z) = c\}$. If $F_z(\mathbf{a}) \neq 0$, then there is a neighborhood U of (a_1, a_2) in $\mathbb{R}^2$, and a C^1-function $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ such that if $(x,y) \in U$ satisfies $F(x, y, f(x,y)) = c$ (i.e., $(x,y,z) \in S$), then $z = f(x,y)$.*

Proof. To prove this, we can actually construct the function $z = f(x,y)$. First, we will need two assumptions, which will result in no loss of generality: 1) Assume $c = 0$, and 2) $F_z(\mathbf{a}) = \frac{\partial F}{\partial z}(a_1, a_2, a_3) > 0$. The first is easy to assume since if $F(\mathbf{a}) = c \neq 0$, you could easily replace F with a function $G = F - c$, resulting in a function with identical properties and $\mathbf{a}$ is on the 0-level set of G. I will get to why the second assumption is useful presently.

Since $F_z(a_1, a_2, a_3) > 0$, it follows by the continuity of the partial derivatives that $F_z(x,y,z) > 0$ in a small neighborhood U of $\mathbf{a}$. Make this neighborhood a small enough closed, cubic δ-set about $\mathbf{a}$:

$$U(\mathbf{a}) = U(a_1, a_2, a_3) = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, |x - a_1| \leq \delta,\ |y - a_2| \leq \delta,\ |z - a_3| \leq \delta \right\}.$$

(The reason for the rectilinear set will make the idea a bit easier.)

Now, since $F_z(a_1, a_2, a_3) > 0$ and $F(a_1, a_2, a_3) = 0$, it follows that for every r where $0 < r \leq \delta$, we have $F(a_1, a_2, a_3 - r) < 0$ and $F(a_1, a_2, a_3 + r) > 0$. You can look at the function $F(a_1, a_2, z)$ as a function of one variable; Call it $F(a_1, a_2, z) = G_{a_1,a_2}(z)$ defined on the interval $z \in [a_3 - \delta, a_3 + \delta]$. Again, since $\frac{\partial F}{\partial z} > 0$, this means that $G'_{a_1,a_2}(z) > 0$ on $z \in [a_3 - \delta, a_3 + \delta]$, and $G_{a_1,a_2}(z)$ is a strictly increasing function. A good way to visualize it is to think that the vertical line (a_1, a_2, z) is cutting through the level sets of F. This is true because the gradient of F on U always has a component in the z-direction. Hence it is never perpendicular to the image of the line (a_1, a_2, z), for $z \in [a_3 - \delta, a_3 + \delta]$. Of course, we have $G_{a_1,a_2}(a_3) = 0$ since $G_{a_1,a_2}(a_3) = F(a_1, a_2, a_3) = 0$ by assumption.

Now, let (x,y) be ANY point, where $x \in [a_1 - \delta, a_1 + \delta]$, and $y \in [a_2 - \delta, a_2 + \delta]$ (these comprise all of the x and y coordinates of the points in $U(\mathbf{a})$ when z is within δ of a_3). Then we can form a new function $G_{x,y}(z) = F(x,y,z)$, again defined on $z \in [a_3 - \delta, a_3 + \delta]$. This function $G_{x,y}(z)$ is continuous in z, strictly increasing and $G_{x,y}(a_3 - \delta) < 0$ and $G_{x,y}(a_3 + \delta) > 0$ (Why is this the case? Think about this one.) By the Intermediate Value Theorem (yes, the one from Calculus I), there must be a unique value of z where $G_{x,y}(z) = 0$. Since this is true for any choice of (x, y) as above, we get an assignment $(x, y) \longrightarrow z$ at all points $(x, y, z) \in U(\mathbf{a})$. This is our desired choice for $z = f(x, y)$.

To see that this choice of f is C^1, consider what its derivative will be: We have the C^1 function

$$F(x, y, f(x, y)) = F(x, y, z(x, y)) = 0.$$

Recall the tangent hyperplane to the level set of a function $F(x, y, z)$ at a point $\mathbf{a} \in \mathbb{R}^3$ is given by $\nabla F \cdot (\mathbf{x} - \mathbf{a}) = 0$. Written out, this means

$$\frac{\partial F}{\partial x}(\mathbf{a})(x - a_1) + \frac{\partial F}{\partial y}(\mathbf{a})(y - a_2) + \frac{\partial F}{\partial z}(\mathbf{a})(z - a_3) = 0.$$

Solving for z, and knowing that $\frac{\partial F}{\partial z}(\mathbf{a}) \neq 0$, we get

$$z = a_3 + \frac{\frac{\partial F}{\partial x}(\mathbf{a})}{-\frac{\partial F}{\partial z}(\mathbf{a})}(x - a_1) + \frac{\frac{\partial F}{\partial y}(\mathbf{a})}{-\frac{\partial F}{\partial z}(\mathbf{a})}(y - a_2),$$

we can equate the quantities to what we know the tangent space to the graph of a function $z = f(x, y)$ would be. Indeed, $a_3 = f(a_1, a_2)$ and

$$z = f(a_1, a_2) + \frac{\partial f}{\partial x}(a_1, a_2)(x - a_1) + \frac{\partial f}{\partial y}(a_1, a_2)(y - a_2).$$

Hence we have at $\mathbf{a}$,

$$\frac{\partial f}{\partial x}(a_1, a_2) = \frac{-\frac{\partial F}{\partial x}(\mathbf{a})}{\frac{\partial F}{\partial z}(\mathbf{a})}, \quad \text{and} \quad \frac{\partial f}{\partial y}(a_1, a_2) = -\frac{\frac{\partial F}{\partial y}(\mathbf{a})}{\frac{\partial F}{\partial z}(\mathbf{a})}.$$

And since this also work for all points on the level set in $U(\mathbf{a})$, the function $z = f(x, y)$ is c^1 on $U(\mathbf{a})$.

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