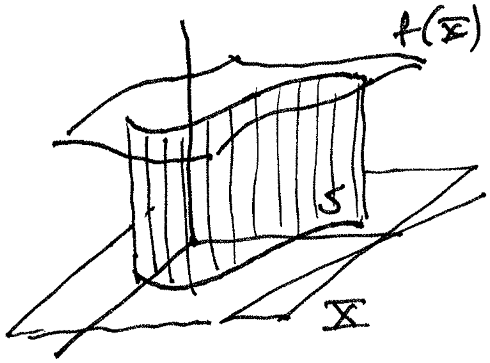
Given a C^1 function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ and $\vec{c} : [a,b] \to \mathbb{R}^n$ a C^1 path (parametrized by t) $\ni$ $\vec{c}([a,b]) \subset X$, The scalar (line) or path integral of f on $\vec{c}$ is

$$\int_{\vec{c}} f\,ds = \int_a^b f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$

Notes ① Here ds is the diff of arc-length, so that the integral of f over $\vec{c}$ is parameter-independent.

② Interpretation: For $f \geq 0$ on $\vec{c}$,



$$\int_{\vec{c}} f\,ds = \text{area of curved (S) surface bounded by } f(\vec{c}([a,b])) \text{ \& } \vec{c}.$$

Given $\vec{c} : [a,b] \to \mathbb{R}^n$ a C^1-path in $X \subset \mathbb{R}^n$, and $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ a vector field, one can see that $\vec{F} \circ \vec{c}(t) = \vec{F}(\vec{c}(t)) : [a,b] \to \mathbb{R}^n$ is another path.

Think of $\vec{F}$ as a force field, and a particle traveling along $\vec{c}$ will be affected by the field.

We can ask for the work done by $\vec{F}$ on the particle.

Straight-line path: Let $\vec{c}$ be linear and $\vec{d}$ its unit vector ~~constant velocity~~. Let $\vec{F}$ be a constant force (vector field).

$$\Longrightarrow \quad W = \vec{F} \cdot \vec{d} = \|\vec{F}\| \, \|\vec{d}\| \cos\theta$$
$$= \|\vec{F}\| \cdot (\text{displacement in direction of } \vec{F})$$

- If $\vec{c}$ is curved, we do this infinitessimally, and

$$\text{Work done by } \vec{F} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

Note again this is a scalar integral (note the dot prod).

- For $\vec{F}$ a vector field, this integral is called a <u>vector</u> <u>line</u> <u>integral</u> of $\vec{F}$ on $\vec{c}$ and can be written

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

Here, displacement along $\vec{c}$ is a vector $d\vec{s}$

$$d\vec{s} = (dx_1, \ldots, dx_n)$$

and is the infinitessimal change in position.

- Since $dx_i = x_i'(t)\,dt$, $d\vec{s} = \vec{c}\,'(t)\,dt$.

- Another interpretation: Assume the particle never stops (i.e., $\vec{c}\,'(t) \neq \vec{0}$ $\forall t \in [a,b]$).

Then
$$\int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt = \int_a^b \vec{F}(\vec{c}(t)) \cdot \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}\|\vec{c}\,'(t)\|\,dt$$
$$= \int_a^b \left(\vec{F}(\vec{c}(t)) \cdot \vec{T}(t)\right) \underbrace{\|\vec{c}\,'(t)\|\,dt}_{\text{scalar}}$$
$$= \int_{\vec{c}} (\vec{F} \cdot \vec{T})\,ds$$

where $\vec{T}(t)$ is the unit tangent vector along $\vec{c}(t)$ @ t.

and $ds = \|d\vec{s}\| = \|\vec{c}\,'(t)\,dt\| = \|\vec{c}\,'(t)\|\,dt$ is the scalar differential.

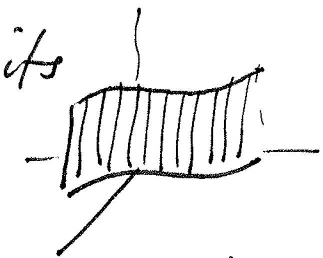
- This is also the path integral or scalar line integral of the tangential component (a scalar fnc of t) of $\vec{F}$ along $\vec{c}$.

(a measure of the boost or hindrance a particle feels by the force while moving along $\vec{c}$).

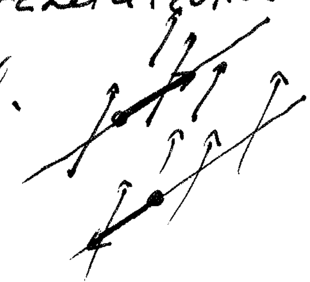
- Note: If $\vec{F}$ is always perp. to path, then
$$\int_{\vec{c}} (\vec{F} \cdot \vec{T})\,ds = 0.$$

Some Facts

Thm 1 A path integral is independent of its parameterization.



Thm 2 A vector line integral depends on the parameterization only in the direction of travel.



Curve Facts

(1) Let $\vec{c}: I \to \mathbb{R}^n$ be a piecewise C^1 path on $I = [a,b]$.
For $h: \overset{=[c,d]}{\cancel{I}} \to J = [c,d]$ a 1-1, onto, C^1 function, w/ C^1 inverse,
$\vec{p}: J \to \mathbb{R}^n$, $\vec{p} = \vec{c} \circ h$ is called a reparameterization.

(2) A curve in $\mathbb{R}^n$ has 2 directions of travel (increasing values for t) (assuming $\vec{c}$ is 1-1).

ex. $x(t) = \sin t$, $y(t) = \sin^2 t$

- direction of travel is called an orientation
- A reparameterization is called orient. preserving if direction is the same.

  else reversing
- A curve is called simple if $\vec{c}: [a,b] \to \mathbb{R}^n$ is injective, except at ends, and closed if $\vec{c}(b) = \vec{c}(a)$.

③ Path integrals are defined on curves and line integrals on oriented curves.
Parametrizations only facilitate the calculation.

2 other facts

(I) For a line integral $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$

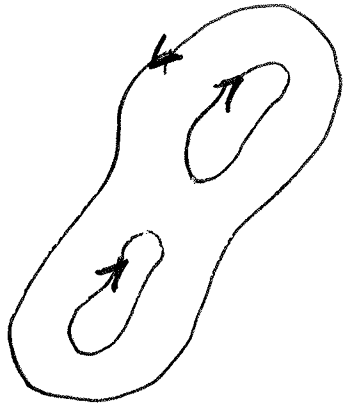
$$\vec{F} \cdot d\vec{s} = F_1\, dx_1 + \cdots + F_n\, dx_n = \sum_{i=1}^{n} F_i\, dx_i$$

is called a differential 1-form.

(II) If $\vec{c}$ is a closed simple path, then notation for a line integral of $\vec{F}$ on $\vec{c}$ is sometimes

$$\oint_{\vec{c}} \vec{F} \cdot d\vec{s}.$$

Thm (Green) Let D be a closed, bounded region in $\mathbb{R}^2$ whose boundary $\vec{C} = \partial D$ is a finite union of simple closed curves, oriented so that D is always on left.



For a C^1-vector field $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ on D, we have

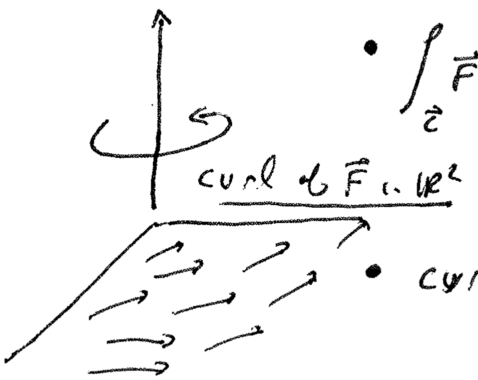
$$\oint_{\vec{C}} \vec{F} \cdot d\vec{s} \overset{(2)}{=} \oint_{\vec{C}} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\overset{(1)}{=} \iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dA$$

Notes [1] (a) is obvious and since $\nabla\times\vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$

so is (b).

[2] The thm says that the vector line integral of F on ∂D equals the curl of F on D:



• $\int_{\vec{c}} \vec{F}\cdot d\vec{s}$ measures how much (in total) the particle along $\vec{c}$ sees or feels $\vec{F}$.

• curl in 2-dim measures the rotation in $\mathbb{R}^2$ one would feel while flowing along $\vec{F}$.

$\Longrightarrow$ sum of the total push or pull of a particle by $\vec{F}$ on ∂D equals the total rotation effect of $\vec{F}$ on D.

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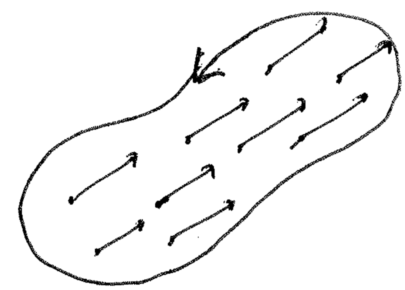
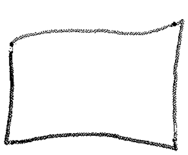
ex. For a constant vector field $\vec{F}$



• $\nabla\times\vec{F} = \vec{0}$, so RHS $= 0$.

• What happens on LHS??



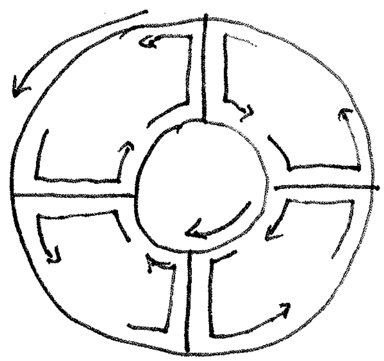
[3] For ~~elementary~~ regions elementary of both types one can show:

Lemma 1 If D is elementary of type 1, then there is an orientation of $\vec{c} = \partial D \ni \int_{\vec{c}} M\,dx = -\iint_D \frac{\partial M}{\partial y}\,dx\,dy$ Green's Thm

Lemma 2 If D is elem of type 2, then this same orientation of $\vec{c}$ yields $\int_{\vec{c}} N\,dy = \iint \frac{\partial N}{\partial x}\,dx\,dy$

Thus Green's Thm holds for D elem of both types.

(14) <u>Fact</u> Any region of Green's Thm is considered "nice": can be cut up into a finite set of regions D_i ∋



① The ends of each cut lie in ∂D

② The cuts do not intersect

③ $D = \sqcup D_i$ and each D_i is elem of both types

④ Each cut intersects exactly 2 D_i's and ~~the boundary of~~ each cut is oriented in the opposite way in these 2 pieces.

Note: The vector line integrals will cancel out along cuts and the double integral won't even see the cuts.