PROOF OF A LEMMA FROM GREEN'S THEOREM

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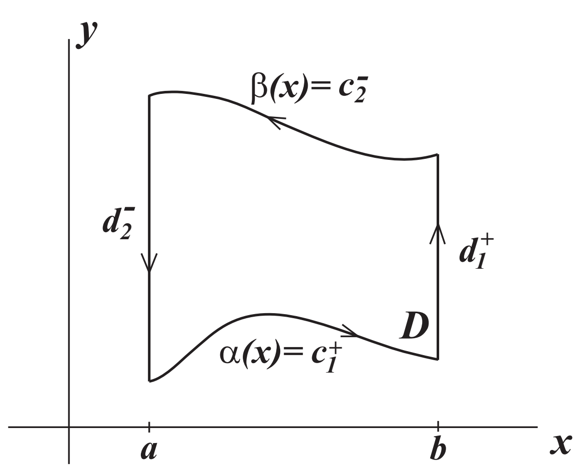
In the discussion following Green's Theorem today, I mentioned a lemma that provides the backbone for the proof of the theorem: That lemma is proved here.

Let D be a closed, bounded region in the plane which is elementary of Type 1 (so that it can be written as a interval in the x-direction and as the difference between two functions of x in the y-direction. Orient $\mathbf{c} = \partial D$ so that (like Green's Theorem) D is always on the left.

Lemma. *If D is as above and $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ is a C^1-vector field on D, then*

$$\oint_{\mathbf{c}} M\,dx = -\iint_D \frac{\partial M}{\partial y}\,dx\,dy. \tag{1}$$



For clarity of notation, the orientation on $\mathbf{c}$ will be denoted

$$\mathbf{c}^+ = \mathbf{c}_1^+ + \mathbf{d}_1^+ + \mathbf{c}_2^- + \mathbf{d}_2^-.$$

The superscripts denote that fact that the orientation of each curve piece is either compatible with (plus sign) or runs counter to (minus sign) the variable which parameterizes the path. For example, on the interval $[a,b]$, x parameterizes both the top and bottom curves, but for the top curve, the orientation runs counter to increasing values of x.

Proof. Neglecting the sign for a minute, the right-hand-side of Equation 1 is

$$\begin{aligned}\iint_D \frac{\partial M}{\partial y}\,dx\,dy &= \int_a^b \int_{\alpha(x)}^{\beta(x)} \frac{\partial M}{\partial y}\,dy\,dx \\ &= \int_a^b \left(M\left(x,\beta(x)\right) - M\left(x,\alpha(x)\right)\right)\,dx,\end{aligned}$$

by the Fundamental Theorem of Calculus in the "inside" integral.

Now we have

$$-\int_a^b M(x,\beta(x))\,dx = \int_{\mathbf{c}_2^-} M\,dx \quad \text{and} \quad \int_a^b M(x,\alpha(x))\,dx = \int_{\mathbf{c}_1^+} M\,dx,$$

while $\int_{\mathbf{d}_1^+} M\,dx$ and $\int_{\mathbf{d}_2^-} M\,dx$ are both 0 (why? x is constant here!). Thus

$$\begin{aligned}\iint_D \frac{\partial M}{\partial y}\,dx\,dy &= \int_a^b \left(M\left(x,\beta(x)\right) - M\left(x,\alpha(x)\right)\right)\,dx \\ &= -\int_{\mathbf{c}_2^-} M\,dx - \int_{\mathbf{c}_1^+} M\,dx = -\oint_{\mathbf{c}^+} M\,dx.\end{aligned}$$

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