THE DIFFERENTIAL

110.211 HONORS MULTIVARIABLE CALCULUS
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Recall that for a variable x, a small change in x is denoted $\Delta x = (x+h) - x = \text{h}$, where h is a number near 0. As the value of h tends to 0, Δx also vanishes. But we can mark the vanishing of Δx via what is called an infinitesimal change in x, and denote it dx, so that

$$\Delta x \xrightarrow{h\to 0} dx.$$

Really, this has meaning mostly in the context of how other quantities change that depend on x. dx is called he differential of x.

Now let $f : X \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function, and $a \in X$. For the graph $y = f(x)$, the quantity

$$\Delta y = \Delta f = f(x + \Delta x) - f(x).$$

As $h \to 0$, Δf tends to $df = dy$. Just how the dependent variable y is changing as one varies x is important in the study of functional relationships between entities, and is the motivation behind the Liebniz notation in calculus $\frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x)$. Note that as an alternate definition, one can call the quantities dx, and dy actual new variables, whose relationship is tied to the relationship between y and x, namely $y = f(x)$. This alternate definition provides a much more concrete foundation for which to use these quantities, but structurally does not change their meaning.

The quantity df is called the differential of f, and represents an infinitesimal change in f given an infinitesimal change in its independent variable x: at $x = a$, we have

$$df = f'(a)dx, \text{ or } df(a) = f'(a)dx$$

to reflect the idea that this differential will change as we vary the point $x = a$.

Some notes:

- This will make more sense later, when we discuss differential forms, but the differential of f, df, is a *differential* 1*-form*.
- This concept embodies the Substitution Rule (the Anti-Chain Rule) in single variable calculus:

$$\int_a^b f\left(g(x)\right) g'(x)\, dx \xrightarrow[du=g'(x)\,dx]{u=g(x)} \int_{g(a)}^{g(b)} f(u)\, du.$$

Indeed, let f be a function of u, so that at $u = \alpha$,

$$df(\alpha) = f'(\alpha)\, du = \left(f'(u)\Big|_{u=\alpha} \right) du.$$

If $u = u(x)$ is also a function of x, we can then write f as a function of x: $f\left(u(x)\right)$. It's differential, then, also varies as x varies. For $u = u(x)$, where $u(a) = \alpha$ for some

a, we have $du = u'(x)\,dx$, and

$$df(\alpha) = \left(f'(u)\Big|_{u=\alpha} \right) du = \left(f'(u)\Big|_{u(a)=\alpha} \right) \left(u'(x)\Big|_{x=a} \right) dx = (f \circ u)'(a)\,dx.$$

Really, the differential here is the differential of the composition, but we can view the differential of f simply as a function of x, and write write

$$df(a) = (f \circ u)'(a)\,dx.$$

In many variables, let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and $\mathbf{a} \in X$. df is the sum of the *partial differential forms* (differential forms in the coordinate directions), $\frac{\partial f}{\partial x_i}\,dx_i$, and

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \ldots + \frac{\partial f}{\partial x_n}\,dx_n = \sum_{n=1}^{n} \frac{\partial f}{\partial x_i}\,dx_i. \tag{1}$$

This quantity represents an infinitesimal change in f in terms of its coordinate changes dx_i.

As a function, $\Delta f = f(\mathbf{a} + \Delta\mathbf{x}) - f(\mathbf{a}) = f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a})$, where $\Delta\mathbf{x} = \mathbf{h}$ is a vector of small changes in each of the coordinate directions. Written out, Δf will contain many terms which are not linear in $\Delta\mathbf{x}$. As $\Delta\mathbf{x}$ tends to 0, only the linear parts of these terms will survive terms die off (the higher-degree terms will die off quickly, leaving only the linear terms). One can then see directly how the differential of a function operates:

Example 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + xy - x - y + \sin x$. Here $\Delta\mathbf{x} = (\Delta x, \Delta y))^T$, and

$$\begin{aligned} \Delta f(\pi, 0) &= f\left((\pi, 0)^T + (\Delta x, \Delta y)^T \right) - f(\pi, 0) \\ &= (\pi + \Delta x)^2 + (\pi + \Delta x)(\Delta y) - (\pi + \Delta x) - \Delta y + \sin(\pi + \Delta x) - \pi^2 + \pi \\ &= \pi^2 + 2\pi\Delta x + (\Delta x)^2 + \pi\Delta y + \Delta x\Delta y - \pi - \Delta x - \Delta y - \sin(\Delta x) - \pi^2 + \pi. \end{aligned}$$

Notice here that all of the terms not containing a Δx or a Δy cancel out. Notice also that for very small values of Δx, the function $\sin(\Delta x) \approx \Delta x$. This is called a first-order approximation of the sine function near $x = 0$ (recall this from single variable calculus. Likewise, for very small values of Δx and Δy, all of the other higher-order terms vanish double fast, leaving only the linear terms:

$$\Delta f(\pi, 0) = (2\pi - 1)\Delta x - \Delta x + (\pi - 1)\Delta y = (2\pi - 2)\Delta x + (\pi - 1)\Delta y.$$

Passing to the infinitesimals, we get $\Delta f \longrightarrow df$, and $\Delta\mathbf{x} \longrightarrow d\mathbf{x} = (dx, dy)^T$, and we get

$$df(\pi, 0) = (2\pi - 2)\,dx + (\pi - 1)\,dy.$$

Now compare this to the direct computation, using Equation 1 above. Here

$$\frac{\partial f}{\partial x}(\pi, 0) = (2x - y - 1 + \cos x)\Big|_{\substack{x=\pi \\ y=0}} = (2\pi - 2)$$

and

$$\frac{\partial f}{\partial y}(\pi, 0) = (x - 1)\Big|_{\substack{x=\pi \\ y=0}} = (\pi - 1).$$

The result is the same.